Differentiation, Mixed Exercise 12

1 $f(x) = 10x^2$

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$
$$= \lim_{h\to 0}\frac{10(x+h)^2-10x^2}{h}$$
$$= \lim_{h\to 0}\frac{10x^2+20xh+10h^2-10x^2}{h}$$
$$= \lim_{h\to 0}\frac{20xh+10h^2}{h}$$
$$= \lim_{h\to 0}\frac{h(20x+10h)}{h}$$
$$= \lim_{h\to 0}(20x+10h)$$

As $h \to 0$, $20x + 10h \to 20x$
So $f'(x) = 20x$

2 a A has coordinates (1, 4).
The y-coordinate of B is
$(1 + \delta x)^3 + 3(1 + \delta x)$
$= 1^3 + 3\delta x + 3(\delta x)^2 + (\delta x)^3 + 3 + 3\delta x$
$= (\delta x)^3 + 3(\delta x)^2 + 6\delta x + 4$
Gradient of AB

$$= \frac{y_2-y_1}{x_2-x_1}$$
$$= \frac{(\delta x)^3+3(\delta x)^2+6\delta x+4-4}{\delta x}$$
$$= \frac{(\delta x)^3+3(\delta x)^2+6\delta x}{\delta x}$$
$$= (\delta x)^2+3\delta x+6$$

b As $\delta x \to 0$, $(\delta x)^2 + 3\delta x + 6 \to 6$
Therefore, the gradient of the curve at point A is 6.

3 $y = 3x^2 + 3 + \dfrac{1}{x^2} = 3x^2 + 3 + x^{-2}$

$$\frac{dy}{dx} = 6x - 2x^{-3} = 6x - \frac{2}{x^3}$$

When $x = 1$, $\dfrac{dy}{dx} = 6\times 1 - \dfrac{2}{1^3}$
$= 4$

When $x = 2$, $\dfrac{dy}{dx} = 6\times 2 - \dfrac{2}{2^3}$
$= 12 - \dfrac{2}{8}$
$= 11\dfrac{3}{4}$

When $x = 3$, $\dfrac{dy}{dx} = 6\times 3 - \dfrac{2}{3^3}$
$= 18 - \dfrac{2}{27}$
$= 17\dfrac{25}{27}$

The gradients at points A, B and C are 4, $11\frac{3}{4}$ and $17\frac{25}{27}$, respectively.

4 $y = 7x^2 - x^3$

$\dfrac{dy}{dx} = 14x - 3x^2$

$\dfrac{dy}{dx} = 16$ when
$14x - 3x^2 = 16$
$3x^2 - 14x + 16 = 0$
$(3x - 8)(x - 2) = 0$
$x = \dfrac{8}{3}$ or $x = 2$

5 $y = x^3 - 11x + 1$

$\dfrac{dy}{dx} = 3x^2 - 11$

$\dfrac{dy}{dx} = 1$ when
$3x^2 - 11 = 1$
$3x^2 = 12$
$x^2 = 4$
$x = \pm 2$

When $x = 2$, $y = 2^3 - 11(2) + 1 = -13$
When $x = -2$, $y = (-2)^3 - 11(-2) + 1 = 15$
The gradient is 1 at the points (2, −13) and (−2, 15).

6 a $f(x) = x + \dfrac{9}{x} = x + 9x^{-1}$

$f'(x) = 1 - 9x^{-2} = 1 - \dfrac{9}{x^2}$

6 b $f'(x) = 0$ when

$$\frac{9}{x^2} = 1$$
$$x^2 = 9$$
$$x = \pm 3$$

7
$$y = 3\sqrt{x} - \frac{4}{\sqrt{x}} = 3x^{\frac{1}{2}} - 4x^{-\frac{1}{2}}$$
$$\frac{dy}{dx} = \frac{3}{2}x^{-\frac{1}{2}} + \frac{4}{2}x^{-\frac{3}{2}}$$
$$= \frac{3}{2\sqrt{x}} + \frac{2}{\left(\sqrt{x}\right)^3}$$
$$= \tfrac{3}{2}x^{-1} + 2x^{-\frac{3}{2}}$$

8 a
$$y = 12x^{\frac{1}{2}} - x^{\frac{3}{2}}$$
$$\frac{dy}{dx} = 12\left(\frac{1}{2}\right)x^{-\frac{1}{2}} - \frac{3}{2}x^{\frac{1}{2}}$$
$$\frac{dy}{dx} = 6x^{-\frac{1}{2}} - \frac{3}{2}x^{\frac{1}{2}}$$
$$= \frac{3}{2}x^{-\frac{1}{2}}(4 - x)$$

b The gradient is zero when $\frac{dy}{dx} = 0$:

$$\frac{3}{2}x^{-\frac{1}{2}}(4 - x) = 0$$
$x = 4$
When $x = 4$, $y = 12 \times 2 - 2^3 = 16$
The gradient is zero at the point with coordinates (4, 16).

9 a
$$\left(x^{\frac{3}{2}} - 1\right)\left(x^{-\frac{1}{2}} + 1\right) = x + x^{\frac{3}{2}} - x^{-\frac{1}{2}} - 1$$

b
$$y = x + x^{\frac{3}{2}} - x^{-\frac{1}{2}} - 1$$
$$\frac{dy}{dx} = 1 + \frac{3}{2}x^{\frac{1}{2}} + \frac{1}{2}x^{-\frac{3}{2}}$$

c When $x = 4$, $\frac{dy}{dx} = 1 + \frac{3}{2} \times 2 + \frac{1}{2} \times \frac{1}{4^{\frac{3}{2}}}$
$$= 1 + 3 + \frac{1}{16}$$
$$= 4\frac{1}{16}$$

10 Let $y = 2x^3 + \sqrt{x} + \dfrac{x^2 + 2x}{x^2}$
$$= 2x^3 + x^{\frac{1}{2}} + \frac{x^2}{x^2} + \frac{2x}{x^2}$$
$$= 2x^3 + x^{\frac{1}{2}} + 1 + 2x^{-1}$$
$$\frac{dy}{dx} = 6x^2 + \frac{1}{2}x^{-\frac{1}{2}} - 2x^{-2}$$
$$= 6x^2 + \frac{1}{2\sqrt{x}} - \frac{2}{x^2}$$

11 The point (1, 2) lies on the curve with equation $y = ax^2 + bx + c$, so
$2 = a + b + c$ **(1)**

The point (2, 1) also lies on the curve, so
$1 = 4a + 2b + c$ **(2)**

(2) − **(1)** gives:
$-1 = 3a + b$ **(3)**

$\frac{dy}{dx} = 2ax + b$
The gradient of the curve is zero at (2, 1), so
$0 = 4a + b$ **(4)**
(4) − **(3)** gives:
$1 = a$
Substituting $a = 1$ into **(3)** gives $b = -4$
Substituting $a = 1$ and $b = -4$ into **(1)** gives $c = 5$
Therefore, $a = 1$, $b = -4$, $c = 5$

12 a $y = x^3 - 5x^2 + 5x + 2$
$$\frac{dy}{dx} = 3x^2 - 10x + 5$$

b i $\frac{dy}{dx} = 2$
$3x^2 - 10x + 5 = 2$
$3x^2 - 10x + 3 = 0$
$(3x - 1)(x - 3) = 0$
$x = \frac{1}{3}$ or 3
$x = 3$ is the coordinate at P,
so $x = \frac{1}{3}$ at Q.

12 b ii $x = 3 \quad y = 27 - 45 + 15 + 2 = -1$

So equation of the tangent is

$y + 1 = 2(x - 3)$

$y = 2x - 7$

iii When $x = 0, y = -7$

and when $y = 0, x = \frac{7}{2}$

So points R and S are $(0, -7)$ and $(\frac{7}{2}, 0)$.

Length of $RS = \sqrt{(-7)^2 + \left(\frac{7}{2}\right)^2}$

$= 7\sqrt{1 + \frac{1}{4}} = \frac{7}{2}\sqrt{5}$

13 $y = \dfrac{8}{x} - x + 3x^2 = 8x^{-1} - x + 3x^2$

$\dfrac{dy}{dx} = -8x^{-2} - 1 + 6x = -\dfrac{8}{x^2} - 1 + 6x$

When $x = 2$, $y = \dfrac{8}{2} - 2 + 3 \times 2^2 = 14$

$\dfrac{dy}{dx} = -\dfrac{8}{4} - 1 + 12 = 9$

The equation of the tangent through the point (2, 14) with gradient 9 is

$y - 14 = 9(x - 2)$

$y = 9x - 18 + 14$

$y = 9x - 4$

The normal at (2, 14) has gradient $-\dfrac{1}{9}$.

So its equation is

$y - 14 = -\dfrac{1}{9}(x - 2)$

$9y + x = 128$

14 a $2y = 3x^3 - 7x^2 + 4x$

$y = \dfrac{3}{2}x^3 - \dfrac{7}{2}x^2 + 2x$

$\dfrac{dy}{dx} = \dfrac{9}{2}x^2 - 7x + 2$

At (0, 0), $x = 0$, gradient of curve is $0 - 0 + 2 = 2$.

Gradient of normal at (0, 0) is $-\dfrac{1}{2}$.

The equation of the normal at (0, 0) is

$y = -\dfrac{1}{2}x$.

At (1, 0), $x = 1$, gradient of curve is

$\dfrac{9}{2} - 7 + 2 = -\dfrac{1}{2}$.

Gradient of normal at (1, 0) is 2.

14 a The equation of the normal at (1, 0) is $y = 2(x - 1)$.

The normals meet when $y = 2x - 2$ and $y = -\dfrac{1}{2}x$:

$2x - 2 = -\dfrac{1}{2}x$

$4x - 4 = -x$

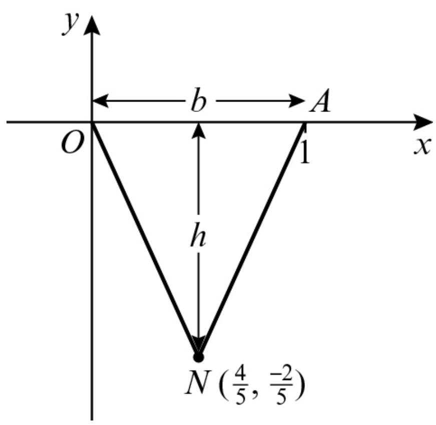
$5x = 4$

$x = \dfrac{4}{5}$

$y = 2\left(\dfrac{4}{5}\right) - 2 = -\dfrac{2}{5}$ $\left(\text{check in } y = -\dfrac{1}{2}x\right)$

N has coordinates $\left(\frac{4}{5}, -\frac{2}{5}\right)$.

b



Area of $\triangle OAN = \dfrac{1}{2}$ base × height

Base (b) = 1

Height (h) = $\dfrac{2}{5}$

Area = $\dfrac{1}{2} \times 1 \times \dfrac{2}{5} = \dfrac{1}{5}$

15 $y = x^3 - 2x^2 - 4x - 1$

When $x = 0$, $y = -1$ so the point P is $(0, -1)$

For the gradient of line L:

$\dfrac{dy}{dx} = 3x^2 - 4x - 4$

At point P, when $x = 0$, $\dfrac{dy}{dx} = -4$

The y-intercept of line L is -1.

Equation of L is $y = -4x - 1$.

Point Q is where the curve and line intersect:

$x^3 - 2x^2 - 4x - 1 = -4x - 1$

$x^3 - 2x^2 = 0$

15 $x^2(x-2)=0$

$x = 0$ or 2

$x = 0$ at point P, so $x = 2$ at point Q.

When $x = 2$, $y = -9$ substituting into the original equation

Using Pythagoras' theorem:

$$\text{distance } PQ = \sqrt{(2-0)^2+(-9-(-1))^2} = \sqrt{68} = \sqrt{4\times 17} = 2\sqrt{17}$$

16 a $y = x^{\frac{3}{2}} + \dfrac{48}{x} \quad (x>0)$

$$\frac{dy}{dx} = \frac{3}{2}x^{\frac{1}{2}} - \frac{48}{x^2}$$

Putting $\dfrac{dy}{dx} = 0$:

$$\frac{3}{2}x^{\frac{1}{2}} = \frac{48}{x^2}$$

$$x^{\frac{5}{2}} = 32$$

$$x = 4$$

Substituting $x = 4$ into $y = x^{\frac{3}{2}} + \dfrac{48}{x}$ gives:

$y = 8 + 12 = 20$

So $x = 4$ and $y = 20$ when $\dfrac{dy}{dx} = 0$.

b $\dfrac{d^2y}{dx^2} = \dfrac{3}{4}x^{-\frac{1}{2}} + \dfrac{96}{x^3}$

When $x = 4$, $\dfrac{d^2y}{dx^2} = \dfrac{3}{8} + \dfrac{96}{64} = \dfrac{15}{8} > 0$

$\therefore$ minimum

17 $y = x^3 - 5x^2 + 7x - 14$

$$\frac{dy}{dx} = 3x^2 - 10x + 7$$

Putting $3x^2 - 10x + 7 = 0$

$(3x-7)(x-1) = 0$

So $x = \dfrac{7}{3}$ or $x = 1$

When $x = \dfrac{7}{3}$,

$$y = \left(\frac{7}{3}\right)^3 - 5\left(\frac{7}{3}\right)^2 + 7\left(\frac{7}{3}\right) - 14 = -\frac{329}{27}$$

17 $y = -12\dfrac{5}{27}$

When $x = 1$,

$y = 1^3 - 5(1)^2 + 7(1) - 14$

$= -11$

So $\left(\frac{7}{3}, -12\frac{5}{27}\right)$ and $(1, -11)$ are stationary points.

18 a $f'(x) = x^2 - 2 + \dfrac{1}{x^2} \quad (x>0)$

$$f''(x) = 2x - \frac{2}{x^3}$$

When $x = 4$, $f''(x) = 8 - \dfrac{2}{64} = 7\dfrac{31}{32}$

b For an increasing function, $f'(x) \geq 0$

$$x^2 - 2 + \frac{1}{x^2} \geq 0$$

$$\left(x - \frac{1}{x}\right)^2 \geq 0$$

This is true for all x, except $x = 1$ (where $f'(1) = 0$).

So the function is an increasing function.

19 $y = x^3 - 6x^2 + 9x$

$$\frac{dy}{dx} = 3x^2 - 12x + 9$$

Putting $3x^2 - 12x + 9 = 0$

$3(x^2 - 4x + 3) = 0$

$3(x-1)(x-3) = 0$

So $x = 1$ or $x = 3$

So there are stationary points when $x = 1$ and $x = 3$.

$$\frac{d^2y}{dx^2} = 6x - 12$$

When $x = 1$, $\dfrac{d^2y}{dx^2} = 6 - 12 = -6 < 0$, so maximum point

When $x = 3$, $\dfrac{d^2y}{dx^2} = 18 - 12 = 6 > 0$, so minimum point

When $x = 1$, $y = 1 - 6 + 9 = 4$

So $(1, 4)$ is a maximum point.

20 a $f(x) = 3x^4 - 8x^3 - 6x^2 + 24x + 20$

$f'(x) = 12x^3 - 24x^2 - 12x + 24$

$= 12(x^3 - 2x^2 - x + 2)$

$= 12(x - 1)(x^2 - x - 2)$

$= 12(x - 1)(x - 2)(x + 1)$

So $x = 1$, $x = 2$ or $x = -1$

$f(1) = 3 - 8 - 6 + 24 + 20$

$= 33$

$f(2) = 3(2)^4 - 8(2)^3 - 6(2)^2 + 24(2) + 20$

$= 28$

$f(-1) = 3 + 8 - 6 - 24 + 20$

$= 1$

So (1, 33), (2, 28) and (−1, 1) are stationary points.

$f''(x) = 36x^2 - 48x - 12$

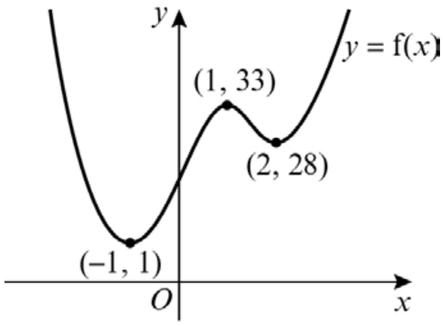
$f''(1) = 36 - 48 - 12 = -24 < 0$, so maximum

$f''(2) = 36(2)^2 - 48(2) - 12 = 36 > 0$, so minimum

$f''(-1)$, $y = 36 + 48 - 12 = 72 > 0$, so minimum

So (1, 33) is a maximum point and (2, 28) and (−1, 1) are minimum points.

b



21 a $f(x) = 200 - \frac{250}{x} - x^2$

$f'(x) = \frac{250}{x^2} - 2x$

b At the maximum point, B, $f'(x) = 0$

$\frac{250}{x^2} - 2x = 0$

$\frac{250}{x^2} = 2x$

$250 = 2x^3$

$x^3 = 125$

$x = 5$

When $x = 5$, $y = f(5) = 200 - \frac{250}{5} - 5^2$

$= 125$

21 b The coordinates of B are (5, 125).

22 a P has coordinates m, $\left(x,\ 5 - \frac{1}{2}x^2\right)$.

$OP^2 = (x - 0)^2 + \left(5 - \frac{1}{2}x^2 - 0\right)^2$

$= x^2 + 25 - 5x^2 + \frac{1}{4}x^4$

$= \frac{1}{4}x^4 - 4x^2 + 25$

b Given $f(x) = \frac{1}{4}x^4 - 4x^2 + 25$

$f'(x) = x^3 - 8x$

When $f'(x) = 0$,

$x^3 - 8x = 0$

$x(x^2 - 8) = 0$

$x = 0$ or $x^2 = 8$

$x = 0$ or $x = \pm 2\sqrt{2}$

c $f''(x) = 3x^2 - 8$

When $x = 0$, $f''(x) = -8 < 0$, so maximum

When $x^2 = 8$, $f''(x) = 3 \times 8 - 8 = 16 > 0$, so minimum

Substituting $x^2 = 8$ into $f(x)$:

$OP^2 = \frac{1}{4} \times 8^2 - 4 \times 8 + 25 = 9$

So $OP = 3$ when $x = \pm 2\sqrt{2}$

23 a $y = 3 + 5x + x^2 - x^3$

Let $y = 0$, then

$3 + 5x + x^2 - x^3 = 0$

$(3 - x)(1 + 2x + x^2) = 0$

$(3 - x)(1 + x)^2 = 0$

$x = 3$ or $x = -1$ when $y = 0$

The curve touches the x-axis at $x = -1$ (A) and cuts the axis at $x = 3$ (C).

C has coordinates (3, 0)

b $\frac{dy}{dx} = 5 + 2x - 3x^2$

Putting $\frac{dy}{dx} = 0$

$5 + 2x - 3x^2 = 0$

$(5 - 3x)(1 + x) = 0$

So $x = \frac{5}{3}$ or $x = -1$

When $x = \frac{5}{3}$,

23 b $y = 3+5\left(\frac{5}{3}\right)+\left(\frac{5}{3}\right)^2-\left(\frac{5}{3}\right)^3 = 9\frac{13}{27}$

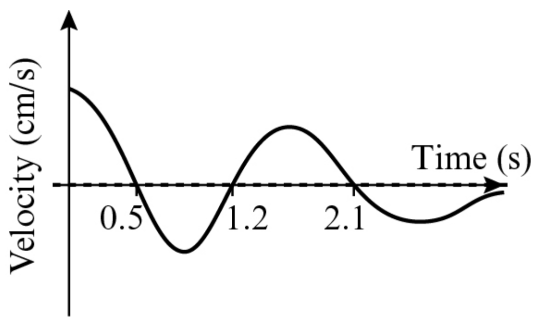
So B is $\left(\frac{5}{3}, 9\frac{13}{27}\right)$.

When $x = -1$, $y = 0$

So A is $(-1, 0)$.

24

x	$y = f(x)$	$y = f'(x)$
$0 < x < 0.5$	Positive gradient	Above x-axis
$x = 0.5$	Maximum	Cuts x-axis
$0.5 < x < 1.2$	Negative gradient	Below x-axis
$x = 1.2$	Minimum	Cuts x-axis
$1.2 < x < 2.1$	Positive gradient	Above x-axis
$x = 2.1$	Maximum	Cuts x-axis
$x > 2.1$	Negative gradient	Below x-axis with asymptote at $y = 0$



25 $V = \pi(40r - r^2 - r^3)$

$\frac{dV}{dr} = 40\pi - 2\pi r - 3\pi r^2$

Putting $\frac{dV}{dr} = 0$

$\pi(40 - 2r - 3r^2) = 0$

$(4 + r)(10 - 3r) = 0$

$r = \frac{10}{3}$ or $r = -4$

As r is positive, $r = \frac{10}{3}$

Substituting into the given expression for V:

$V = \pi\left(40\times\frac{10}{3}-\frac{100}{9}-\frac{1000}{27}\right) = \frac{2300}{27}\pi$

26 $A = 2\pi x^2 + \frac{2000}{x} = 2\pi x^2 + 2000x^{-1}$

$\frac{dA}{dx} = 4\pi x - 2000x^{-2} = 4\pi x - \frac{2000}{x^2}$

Putting $\frac{dA}{dx} = 0$

$4\pi x = \frac{2000}{x^2}$

$x^3 = \frac{2000}{4\pi} = \frac{500}{\pi}$

27 a The total length of wire is

$\left(2y + x + \frac{\pi x}{2}\right)$ m

As total length is 2 m,

$2y + x\left(1+\frac{\pi}{2}\right) = 2$

$y = 1 - \frac{1}{2}x\left(1+\frac{\pi}{2}\right)$

b Area, $R = xy + \frac{1}{2}\pi\left(\frac{x}{2}\right)^2$

Substituting $y = 1-\frac{1}{2}x\left(1+\frac{\pi}{2}\right)$ gives:

$R = x\left(1-\frac{1}{2}x-\frac{\pi}{4}x\right)+\frac{\pi}{8}x^2$

$= \frac{x}{8}(8 - 4x - 2\pi x + \pi x)$

$= \frac{x}{8}(8 - 4x - \pi x)$

c For maximum R, $\frac{dR}{dx} = 0$

$R = x - \frac{1}{2}x^2 - \frac{\pi}{8}x^2$

$\frac{dR}{dx} = 1 - x - \frac{\pi}{4}x$

Putting $\frac{dR}{dx} = 0$

$x = \frac{1}{1+\frac{\pi}{4}}$

$= \frac{4}{4+\pi}$

27 c Substituting $x = \dfrac{4}{4+\pi}$ into R:

$$R = \frac{1}{2(4+\pi)}\left(8 - \frac{16}{4+\pi} - \frac{4\pi}{4+\pi}\right)$$

$$R = \frac{1}{2(4+\pi)} \times \frac{32 + 8\pi - 16 - 4\pi}{4+\pi}$$

$$= \frac{1}{2(4+\pi)} \times \frac{16 + 4\pi}{4+\pi}$$

$$= \frac{4(4+\pi)}{2(4+\pi)^2}$$

$$= \frac{2}{4+\pi}$$

28 a Let the height of the tin be h cm.
The area of the curved surface of the tin = $2\pi xh$ cm^2
The area of the base of the tin = πx^2 cm^2
The area of the curved surface of the lid = $2\pi x$ cm^2
The area of the top of the lid = πx^2 cm^2
Total area of sheet metal is 80π cm^2.
So $2\pi x^2 + 2\pi x + 2\pi xh = 80\pi$

$$h = \frac{40 - x - x^2}{x}$$

The volume, V, of the tin is given by
$V = \pi x^2 h$

$$= \frac{\pi x^2(40 - x - x^2)}{x}$$

$$= \pi(40x - x^2 - x^3)$$

b $\dfrac{dV}{dx} = \pi(40 - 2x - 3x^2)$

Putting $\dfrac{dV}{dx} = 0$

$40 - 2x - 3x^2 = 0$
$(10 - 3x)(4 + x) = 0$

So $x = \dfrac{10}{3}$ or $x = -4$

But x is positive, so $x = \dfrac{10}{3}$

c $\dfrac{d^2V}{dx^2} = \pi(-2 - 6x)$

When $x = \dfrac{10}{3}$, $\dfrac{d^2V}{dx^2} = \pi(-2 - 20) < 0$
So V is a maximum.

28 d $$V = \pi\left(40 \times \frac{10}{3} - \left(\frac{10}{3}\right)^2 - \left(\frac{10}{3}\right)^3\right)$$

$$= \pi\left(\frac{400}{3} - \frac{100}{9} - \frac{1000}{27}\right)$$

$$= \frac{2300}{27}\pi$$

e Lid has surface area $\pi x^2 + 2\pi x$

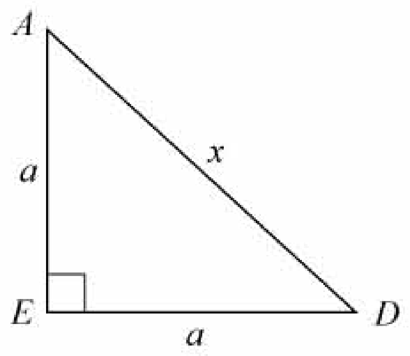
When $x = \dfrac{10}{3}$,

this is $\pi\left(\dfrac{100}{9} + \dfrac{20}{3}\right) = \dfrac{160}{9}\pi$

Percentage of total surface area =

$$\frac{\frac{160}{9}\pi}{80\pi} \times 100 = \frac{200}{9} = 22.2...\%$$

29 a Let the equal sides of ΔADE be a metres.



Using Pythagoras' theorem,
$a^2 + a^2 = x^2$
$2a^2 = x^2$

$$a^2 = \frac{x^2}{2}$$

Area of $\Delta ADE = \dfrac{1}{2} \times$ base $\times$ height

$$= \frac{1}{2} \times a \times a$$

$$= \frac{x^2}{4}\text{ m}^2$$

b Area of two triangular ends

$$= 2 \times \frac{x^2}{4} = \frac{x^2}{2}$$

Let the length $AB = CD = y$ metres

29 b Area of two rectangular sides

$= 2 \times ay = 2ay = 2\sqrt{\frac{x^2}{2}}y$

So $S = \frac{x^2}{2} + 2\sqrt{\frac{x^2}{2}}y = \frac{x^2}{2} + xy\sqrt{2}$

But capacity of storage tank $= \frac{1}{4}x^2 \times y$

So $\frac{1}{4}x^2y = 4000$

$y = \frac{16\,000}{x^2}$

Substituting for y in equation for S gives:

$S = \frac{x^2}{2} + \frac{16\,000\sqrt{2}}{x}$

c $\frac{dS}{dx} = x - \frac{16\,000\sqrt{2}}{x^2}$

Putting $\frac{dS}{dx} = 0$

$x = \frac{16\,000\sqrt{2}}{x^2}$

$x^3 = 16\,000\sqrt{2}$

$x = 20\sqrt{2} = 28.28$ (4 s.f.)

When $x = 20\sqrt{2}$,

$S = 400 + 800 = 1200$

d $\frac{d^2S}{dx^2} = 1 + \frac{32\,000\sqrt{2}}{x^3}$

When $x = 20\sqrt{2}$, $\frac{d^2S}{dx^2} = 3 > 0$, so value is a minimum.

Challenge

a $(x+h)^7 = x^7 + \binom{7}{1}x^6h + \binom{7}{2}x^5h^2 + \binom{7}{3}x^4h^3 + \ldots$

$= x^7 + 7x^6h + 21x^5h^2 + 35x^4h^3 + \ldots$

b $\frac{\mathrm{d}(x^7)}{\mathrm{d}x} = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

$= \lim_{h\to 0}\frac{(x+h)^7 - x^7}{h}$

$= \lim_{h\to 0}\frac{x^7 + 7x^6h + 21x^5h^2 + 35x^4h^3 - x^7}{h}$

$= \lim_{h\to 0}\frac{7x^6h + 21x^5h^2 + 35x^4h^3}{h}$

$= \lim_{h\to 0}\frac{h(7x^6 + 21x^5h + 35x^4h^2)}{h}$

$= \lim_{h\to 0}(7x^6 + 21x^5h + 35x^4h^2)$

As $h \to 0$, $7x^6 + 21x^5h + 35x^4h^2 \to 7x^6$, so $\frac{\mathrm{d}(x^7)}{\mathrm{d}x} = 7x^6$