Product Rule For Differentiation

Let u and v be functions of x

Find $\frac{d}{dx}(uv)$

Let $y = uv$ (1)

For a small change in x say δx we will have a small change in y, u and v say δy, δu and δv

$$y + \delta y = (u + \delta u)(v + \delta v) \quad (2)$$

(2) - (1)
$$\delta y = (u + \delta u)(v + \delta v) - uv$$

$$\delta y = \cancel{uv} + v\delta u + u\delta v + \delta u \delta v - \cancel{uv}$$

$$\frac{\delta y}{\delta x} = v\frac{\delta u}{\delta x} + u\frac{\delta v}{\delta x} + \delta u\frac{\delta v}{\delta x}$$

Letting $\delta x \to 0$
$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx} + 0\frac{dv}{dx}$$

$$\therefore \quad \frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

In plain English, the differential of the product of two functions in x is given by

"The first times the differential of the second plus the second times the differential of the first."

Example $\frac{d}{dx} e^x \sin x = e^x \cos x + \sin x \times e^x$

$$= e^x (\cos x + \sin x)$$

Quotient Rule For Differentiation

Let u and v be functions of x

Find $\frac{d}{dx}\left(\frac{u}{v}\right)$ Let $y = \frac{u}{v}$ ①

A small change in x say δx will cause small changes in y, u and v say δy, δu and δv

$$y + \delta y = \frac{u + \delta u}{v + \delta v} \quad ②$$

② - ①

$$y = \frac{u + \delta u}{v + \delta v} - \frac{u}{v}$$

$$y = \frac{v(u + \delta u) - u(v + \delta v)}{v(v + \delta v)}$$

$$y = \frac{\cancel{vu} + v\delta u - \cancel{uv} - u\delta v}{v(v + \delta v)}$$

$$y = \frac{v\,\delta u - u\,\delta v}{v(v+\delta v)}$$

$$\frac{\delta y}{dx} = \frac{v\frac{\delta u}{\delta x} - u\frac{\delta v}{\delta x}}{v(v+\delta v)}$$

Letting $\delta x \to 0$ $$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v(v+0)}$$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

In plain English. The differential of a quotient is given by:

The bottom times the differential of the top minus the top times the differential of the bottom all over the bottom squared.

Example $$\frac{d}{dx}\tan x = \frac{d}{dx}\frac{\sin x}{\cos x}$$

$$= \frac{\cos x \times \cos x - \sin x(-\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x} = \sec^2 x$$

Exercise 3D

1d) $\frac{d}{dx} 3x^5(5x-1)^{-1}$

$$= 3x^5 \times (-1)(5x-1)^{-2}(5) + (5x-1)^{-1} \times 15x^4$$

$$= -15x^5(5x-1)^{-2} + 15x^4(5x-1)^{-1}$$

$$= -15x^5(5x-1)^{-2} + 15x^4(5x-1)^{-2}(5x-1)$$

$$= 15x^4(5x-1)^{-2}\left[-x + 5x - 1\right]$$

$$= 15x^4(5x-1)^{-2}(4x-1)$$