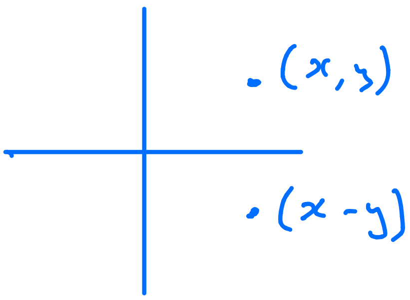
Transformations

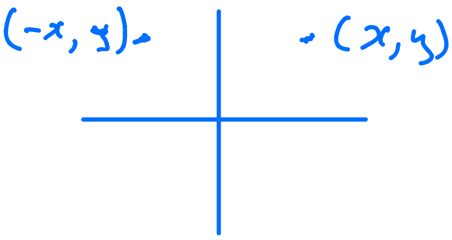
Any 2×2 matrix can represent a transformation in the xy-plane

Reflection in x-axis



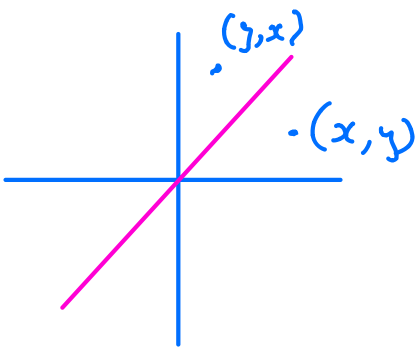
$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ -y \end{pmatrix}$$

Reflection in y-axis



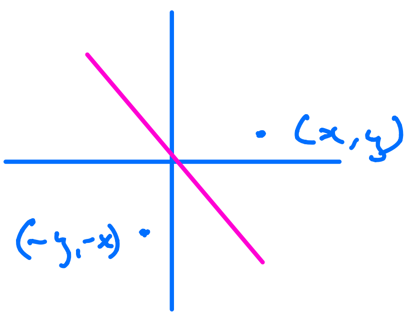
$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x \\ y \end{pmatrix}$$

Reflection in line $y=x$



$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ x \end{pmatrix}$$

Reflection in line $y=-x$



$$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ -x \end{pmatrix}$$

Consider

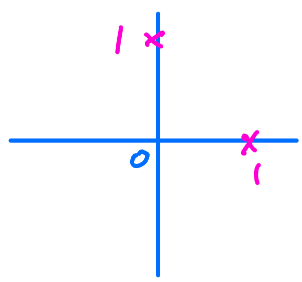
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} b \\ d \end{pmatrix}$$

So if $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a transformation matrix

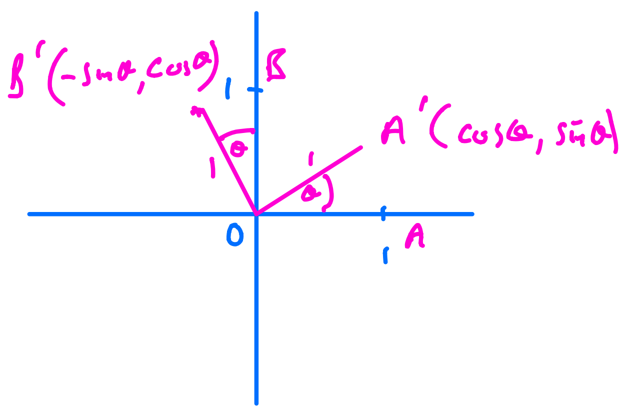
the first column indicates the image of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

the second colun indicates the image of $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$



This is particularly useful for rotations

Consider a rotation of $\theta°$ anti-clockwise which is considered the positive direction



If $A(1,0)$

$A'(\cos\theta, \sin\theta)$

Using argument above the transformation matrix for a rotation of θ° anti-clockwise about origin is given by

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Find transformation matrices for anti-clockwise rotations for

$$30^\circ = \begin{pmatrix} \cos 30 & -\sin 30 \\ \sin 30 & \cos 30 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$$

$$45^\circ = \begin{pmatrix} \cos 45 & -\sin 45 \\ \sin 45 & \cos 45 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

$$60^\circ = \begin{pmatrix} \cos 60 & -\sin 60 \\ \sin 60 & \cos 60 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$$

Find a transformation matrix for a reflection in the x-axis followed by a rotation of 60° anti-clockwise

Reflect in x-axis $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

rotate 60° anti-clockwise $\begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$

Composite Reflect followed by Rotate

$$= \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}$$

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9 You are given the matrix $\mathbf{M} = \begin{pmatrix} 0.8 & 0.6 \\ 0.6 & -0.8 \end{pmatrix}$.

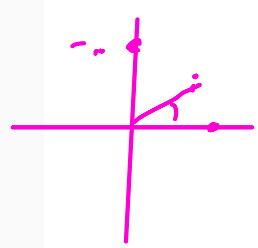
(i) Calculate $\mathbf{M}^2$. [1]

You are now given that the matrix **M** represents a reflection in a line through the origin.

(ii) Explain how your answer to part **(i)** relates to this information. [1]

(iii) By investigating the invariant points of the reflection, find the equation of the mirror line. [3]

(iv) Describe fully the transformation represented by the matrix $\mathbf{P} = \begin{pmatrix} 0.8 & -0.6 \\ 0.6 & 0.8 \end{pmatrix}$. [2]



(v) A composite transformation is formed by the transformation represented by **P** followed by the transformation represented by **M**. Find the single matrix that represents this composite transformation. [2]

(vi) The composite transformation described in part **(v)** is equivalent to a single reflection. What is the equation of the mirror line of this reflection? [1]

i) $\underline{M}^2 = \begin{pmatrix} 0.8 & 0.6 \\ 0.6 & -0.8 \end{pmatrix}\begin{pmatrix} 0.8 & 0.6 \\ 0.6 & -0.8 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

ii) Reflecting in the same line twice returns points to initial positions which is why the resultant transformation is the identity

iii) $\begin{pmatrix} 0.8 & 0.6 \\ 0.6 & -0.8 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$

$0.8x + 0.6y = x$

$0.6x - 0.8y = y$

$\times 5$

$4x + 3y = 5x \Rightarrow 3y = x$ or $y = \frac{1}{3}x$

$3x - 4y = 5y \Rightarrow 9y = 3x$ or $y = \frac{1}{3}x$

so line is $y = \frac{1}{3}x$

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iv) $\begin{pmatrix} 0.8 & -0.6 \\ 0.6 & 0.8 \end{pmatrix}$

rotation anti-clockwise about $(0,0)$

by $\cos^{-1}(0.8)$

about $36.9°$
