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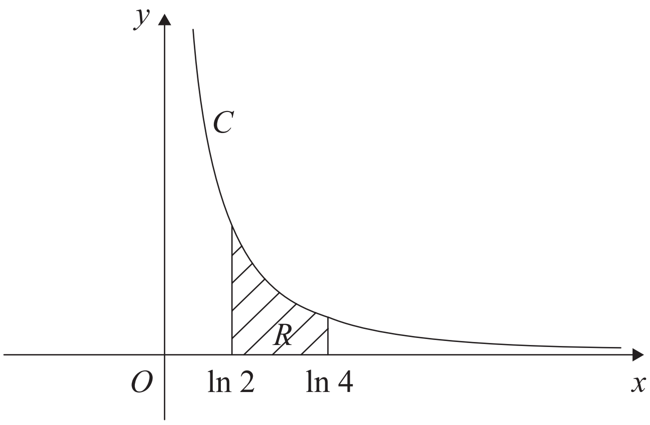

Figure 3

The curve C has parametric equations

$$x = \ln(t+2), \qquad y = \frac{1}{(t+1)}, \qquad t > -1.$$

The finite region R between the curve C and the x-axis, bounded by the lines with equations $x = \ln 2$ and $x = \ln 4$, is shown shaded in Figure 3.

(a) Show that the area of R is given by the integral

$$\int_0^2 \frac{1}{(t+1)(t+2)}\,\mathrm{d}t.$$

(4)

(b) Hence find an exact value for this area.

(6)

(c) Find a cartesian equation of the curve C, in the form $y = \mathrm{f}(x)$.

(4)

(d) State the domain of values for x for this curve.

(1)

a) $\text{Area} = \int_{\ln 2}^{\ln 4} y\,dx$ when $x = \ln 4$, $t = 2$

$= \int_0^2 y \frac{dx}{dt}\,dt$ when $x = \ln 2$, $t = 0$

$= \int_0^2 \frac{1}{(t+1)} \times \frac{1}{(t+2)}\,dt = \int_0^2 \frac{1}{(t+1)(t+2)}\,dt$

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Question 7 continued

b) $$\frac{1}{(t+1)(t+2)} \equiv \frac{1}{t+1} - \frac{1}{t+2}$$

$$\int_0^2 \left(\frac{1}{t+1} - \frac{1}{t+2} \right) dt = \Big[\ln(t+1) - \ln(t+2) \Big]_0^2$$

$$= \left[\ln\left(\frac{t+1}{t+2}\right) \right]_0^2$$

$$= \ln\left(\frac{3}{4}\right) - \ln\left(\frac{1}{2}\right)$$

$$= \ln\left(\frac{\frac{3}{4}}{\frac{1}{2}}\right)$$

$$= \ln \frac{3}{2}$$

c) $$x = \ln(t+2) \Rightarrow e^x = t+2$$

$$e^x - 2 = t$$

$$y = \frac{1}{t+1} \quad \therefore \quad y = \frac{1}{e^x - 1}$$

d) Domain $x > 0$

8.

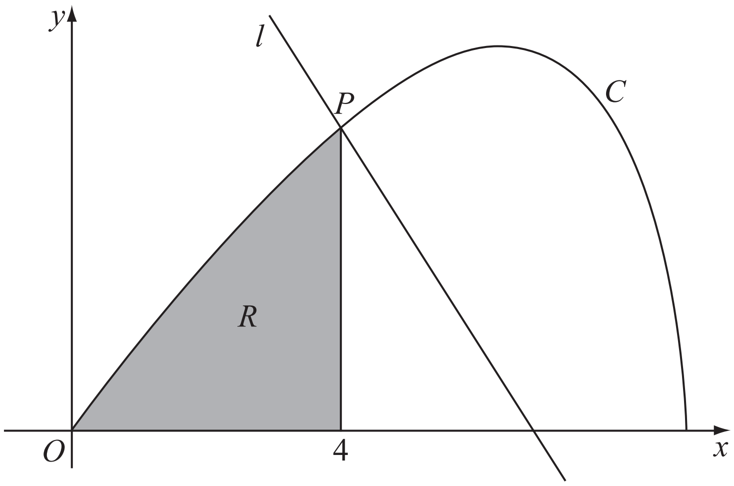

Figure 3

Figure 3 shows the curve C with parametric equations

$$x = 8\cos t, \qquad y = 4\sin 2t, \qquad 0 \leqslant t \leqslant \frac{\pi}{2}.$$

The point P lies on C and has coordinates $(4, 2\sqrt{3})$.

(a) Find the value of t at the point P.

(2)

The line l is a normal to C at P.

(b) Show that an equation for l is $y = -x\sqrt{3} + 6\sqrt{3}$.

(6)

The finite region R is enclosed by the curve C, the x-axis and the line $x = 4$, as shown shaded in Figure 3.

(c) Show that the area of R is given by the integral $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} 64 \sin^2 t \cos t \, dt$.

(4)

(d) Use this integral to find the area of R, giving your answer in the form $a + b\sqrt{3}$, where a and b are constants to be determined.

(4)

a) At P $\quad 4 = 8\cos t$

$\frac{1}{2} = \cos t \qquad \Rightarrow t = \frac{\pi}{3}$

b) $x = 8\cos t \qquad y = 4\sin 2t$

$\frac{dx}{dt} = -8\sin t \qquad \frac{dy}{dt} = 8\cos 2t$

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Question 8 continued

$$\frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt} = -\frac{\cos 2t}{\sin t}$$

At P, $t = \frac{\pi}{3}$ $\Rightarrow$ $\frac{dy}{dx} = \frac{-\cos\frac{2\pi}{3}}{\sin\frac{\pi}{3}}$

$$= \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{3}}$$

Gradient of normal $= -\sqrt{3}$

$P(4, 2\sqrt{3})$

$$y - y_1 = m(x - x_1)$$

$$y - 2\sqrt{3} = -\sqrt{3}(x - 4)$$

$$y - 2\sqrt{3} = -\sqrt{3}x + 4\sqrt{3}$$

$$y = -\sqrt{3}x + 6\sqrt{3}$$

c) Area $= \int_0^4 y\,dx$ $\qquad x = 0,\ t = \frac{\pi}{2}$

$x = 4,\ t = \frac{\pi}{3}$

$$= \int_{\frac{\pi}{2}}^{\frac{\pi}{3}} y\frac{dx}{dt}\,dt = \int_{\frac{\pi}{2}}^{\frac{\pi}{3}} 4\sin 2t(-8\sin t)\,dt$$

$$= -\int_{\frac{\pi}{2}}^{\frac{\pi}{3}} 32\sin 2t \sin t\,dt$$

Q8

(Total 16 marks)

TOTAL FOR PAPER: 75 MARKS

END

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$$= \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} 32(2\sin t\cos t)\sin t \, dt$$

$$= \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} 64\sin^2 t\cos t \, dt$$

d)

Let $u = \sin t$

$$\frac{du}{dt} = \cos t$$

$$du = \cos t \, dt$$

$t = \frac{\pi}{2}$, $u = 1$

$t = \frac{\pi}{3}$, $u = \frac{\sqrt{3}}{2}$

$$= 64\int_{\frac{\sqrt{3}}{2}}^{1} u^2 \, du$$

$$= 64\left[\frac{u^3}{3}\right]_{\frac{\sqrt{3}}{2}}^{1}$$

$$= 64\left[\frac{1}{3} - \frac{\frac{3\sqrt{3}}{8}}{3}\right]$$

$$= 64\left[\frac{1}{3} - \frac{\sqrt{3}}{8}\right]$$

$$\frac{64}{3} - 8\sqrt{3}$$

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7.

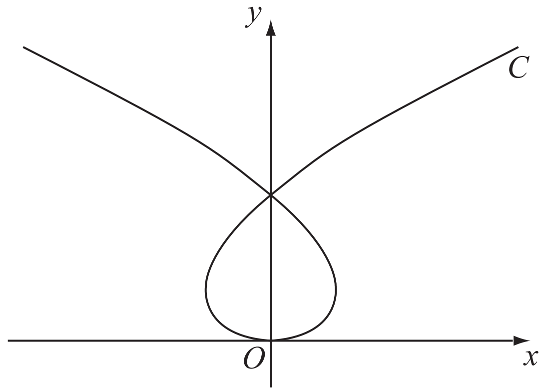

Figure 3

The curve C shown in Figure 3 has parametric equations

$$x = t^3 - 8t, \quad y = t^2$$

where t is a parameter. Given that the point A has parameter $t = -1$,

(a) find the coordinates of A.

(1)

The line l is the tangent to C at A.

(b) Show that an equation for l is $2x - 5y - 9 = 0$.

(5)

The line l also intersects the curve at the point B.

(c) Find the coordinates of B.

(6)

a) $t = -1$, $\quad x = (-1)^3 - 8(-1) = 7$

$y = (-1)^2 = 1$

$\therefore A(7, 1)$

b) $x = t^3 - 8t \qquad y = t^2$

$\frac{dx}{dt} = 3t^2 - 8 \qquad \frac{dy}{dt} = 2t$

$\frac{dy}{dx} = \frac{dy}{dt} / \frac{dx}{dt} = \frac{2t}{3t^2 - 8}$

Leave blank

Question 7 continued

At A, $t=-1$, $\frac{dy}{dx} = \frac{2(-1)}{3(-1)^2-8} = \frac{-2}{-5} = \frac{2}{5}$

Eqn of tgt at A

$y - y_1 = m(x - x_1)$

$y - 1 = \frac{2}{5}(x-7)$

$5y - 5 = 2x - 14$

$0 = 2x - 5y - 9$

c) If on curve $0 = 2(t^3 - 8t) - 5t^2 - 9$

$0 = 2t^3 - 16t - 5t^2 - 9$

$$\begin{array}{r|l} & 2t^2 - 7t - 9 \\ t+1 & 2t^3 - 5t^2 - 16t - 9 \\ & 2t^3 + 2t^2 \\ & -7t^2 - 16t \\ & -7t^2 - 7t \\ & -9t - 9 \\ & -9t - 9 \end{array}$$

$(t+1)(2t^2 - 7t - 9) = 0$

$(t+1)(2t-9)(t+1) = 0$

Other point of intersection when $2t - 9 = 0$

$t = \frac{9}{2}$

$x = \left(\frac{9}{2}\right)^3 - 8\left(\frac{9}{2}\right) =$

$y = \left(\frac{9}{2}\right)^2 = \frac{81}{4}$ $\therefore B\left(\frac{441}{8}, \frac{81}{4}\right)$

Q7

(Total 12 marks)

TOTAL FOR PAPER: 75 MARKS

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8. (a) Using the identity $\cos 2\theta = 1 - 2\sin^2\theta$, find $\int \sin^2\theta \, d\theta$.

(2)

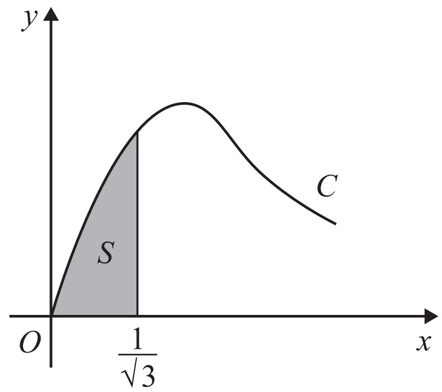

Figure 4

Figure 4 shows part of the curve C with parametric equations

$$x = \tan\theta, \qquad y = 2\sin 2\theta, \qquad 0 \leqslant \theta < \frac{\pi}{2}$$

The finite shaded region S shown in Figure 4 is bounded by C, the line $x = \frac{1}{\sqrt{3}}$ and the x-axis. This shaded region is rotated through 2π radians about the x-axis to form a solid of revolution.

(b) Show that the volume of the solid of revolution formed is given by the integral

$$k\int_0^{\frac{\pi}{6}} \sin^2\theta \, d\theta$$

where k is a constant.

(5)

(c) Hence find the exact value for this volume, giving your answer in the form $p\pi^2 + q\pi\sqrt{3}$, where p and q are constants.

(3)

a) $\cos 2\theta = 1 - 2\sin^2\theta$

$$\int \sin^2\theta \, d\theta = \int \frac{1 - \cos 2\theta}{2} \, d\theta$$

$$= \frac{\theta}{2} - \frac{\sin 2\theta}{4} + c$$

Leave blank

Question 8 continued

b)

$$\text{Volume} = \pi \int_0^{\frac{1}{\sqrt{3}}} y^2 \, dx \qquad x = \frac{1}{\sqrt{3}} \quad \theta = \frac{\pi}{6} \qquad x = 0 \quad \theta = 0$$

$$= \pi \int_0^{\frac{\pi}{6}} y^2 \frac{dx}{d\theta} \, d\theta$$

$$= \pi \int_0^{\frac{\pi}{6}} 4 \sin^2 2\theta \sec^2\theta \, d\theta$$

$$= \pi \int_0^{\frac{\pi}{6}} \frac{4(2\sin\theta\cos\theta)^2}{\cos^2\theta} \, d\theta$$

$$= \pi \int_0^{\frac{\pi}{6}} \frac{4(4\cos^2\theta \sin^2\theta)}{\cos^2\theta} \, d\theta$$

$$= 16\pi \int_0^{\frac{\pi}{6}} \sin^2\theta \, d\theta$$

c)

$$= 16\pi \left[\frac{\theta}{2} - \frac{\sin 2\theta}{4} \right]_0^{\frac{\pi}{6}}$$

$$= 16\pi \left[\frac{\pi}{12} - \frac{\sin \frac{\pi}{3}}{4} - (0 - 0) \right]$$

$$= 16\pi \left[\frac{\pi}{12} - \frac{\sqrt{3}}{8} \right]$$

$$= \frac{4\pi^2}{3} - 2\pi\sqrt{3}$$

Q8

(Total 10 marks)

TOTAL FOR PAPER: 75 MARKS

END

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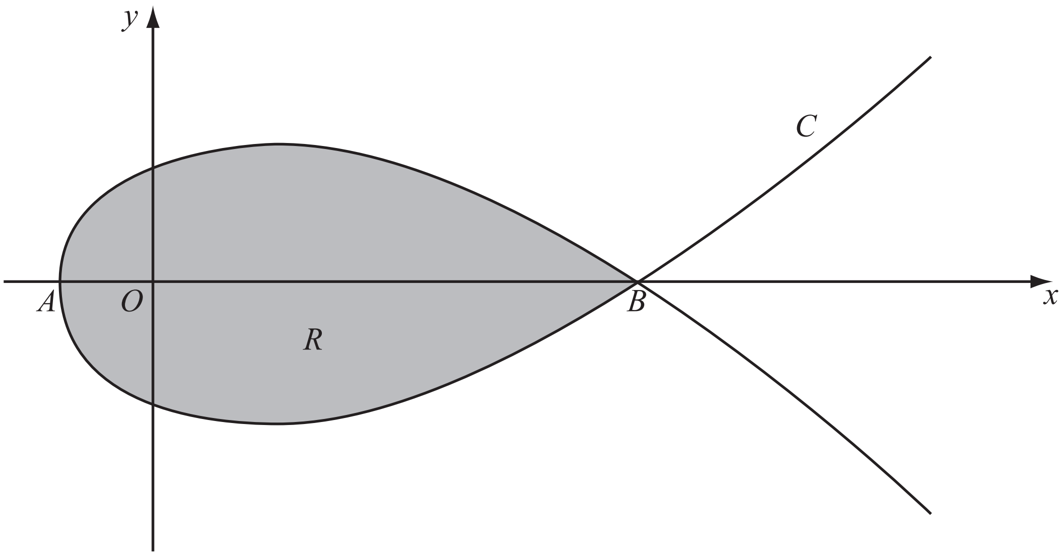

Figure 2

Figure 2 shows a sketch of the curve C with parametric equations

$$x = 5t^2 - 4, \quad y = t(9 - t^2)$$

The curve C cuts the x-axis at the points A and B.

(a) Find the x-coordinate at the point A and the x-coordinate at the point B.

(3)

The region R, as shown shaded in Figure 2, is enclosed by the loop of the curve.

(b) Use integration to find the area of R.

(6)

a) At A and B, $y = 0$ $\quad t(9 - t^2) = 0$

$\Rightarrow t = 0, 3, -3$

When $t = 0$, $x = 5(0)^2 - 4 = -4$

When $t = \pm 3$ $x = 5(\pm 3)^2 - 4 = 41$

x-coord at A $= -4$

x-coord at B $= 41$

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Question 7 continued

b) $$\text{Area} = 2\int_{-4}^{41} |y|\,dx = 2\int_{0}^{3} y\frac{dx}{dt}\,dt$$

$$= 2\int_{0}^{3} t(9-t^2)(10t)\,dt$$

$$= 2\int_{0}^{3} (90t^2 - 10t^4)\,dt$$

$$= 2\left[30t^3 - 2t^5\right]_0^3$$

$$= 2\left[30(3)^3 - 2(3)^5 - (0-0)\right]$$

$$= 648 \text{ units}^2$$

Leave blank

4. A curve C has parametric equations

$$x = \sin^2 t, \quad y = 2\tan t, \quad 0 \leqslant t < \frac{\pi}{2}$$

(a) Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of t.

(4)

The tangent to C at the point where $t = \frac{\pi}{3}$ cuts the x-axis at the point P.

(b) Find the x-coordinate of P.

(6)

a) $\frac{dx}{dt} = 2\sin t\cos t \qquad \frac{dy}{dt} = 2\sec^2 t$

$\frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt} = \frac{2\sec^2 t}{2\sin t\cos t}$

$= \frac{1}{\sin t\cos^3 t}$

b) When $t = \frac{\pi}{3}$, $\frac{dy}{dx} = \frac{1}{\sin\frac{\pi}{3}\cos^3\frac{\pi}{3}} = \frac{1}{\frac{\sqrt{3}}{2} \times \left(\frac{1}{2}\right)^3}$

$= \frac{16}{\sqrt{3}}$

When $t = \frac{\pi}{3}$, $x = \sin^2\frac{\pi}{3} = \frac{3}{4}$

$y = 2\tan t = 2\sqrt{3}$

tangent at C

$y - y_1 = m(x - x_1)$

$y - 2\sqrt{3} = \frac{16}{\sqrt{3}}\left(x - \frac{3}{4}\right)$

Leave blank

Question 4 continued

$$\sqrt{3}y - 6 = 16x - 12$$

$$16x - \sqrt{3}y - 6 = 0$$

Cuts x-axis when $y = 0$

$$16x - 0 - 6 = 0$$

$$x = \frac{6}{16}$$

$$x = \frac{3}{8}$$

x-coord of P $= \frac{3}{8}$