

9.

Figure 3

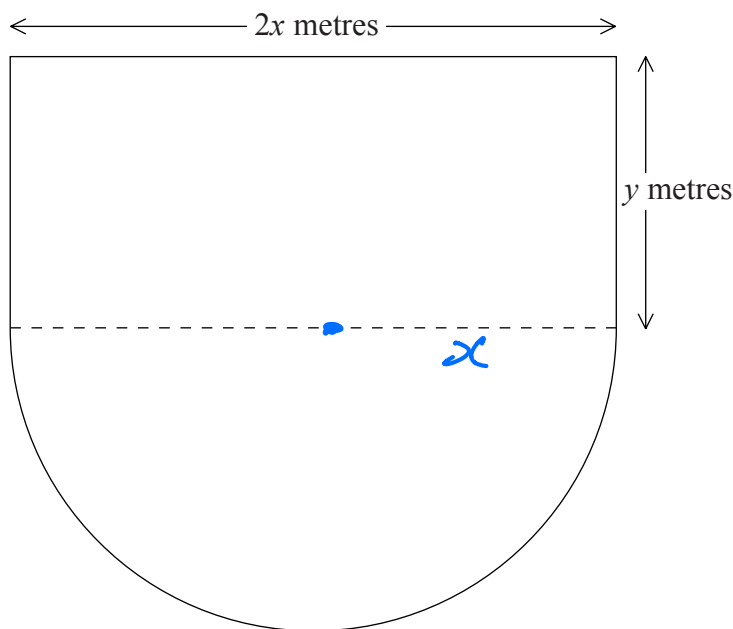


Figure 3 shows the plan of a stage in the shape of a rectangle joined to a semicircle. The length of the rectangular part is $2x$ metres and the width is y metres. The diameter of the semicircular part is $2x$ metres. The perimeter of the stage is 80 m.

- (a) Show that the area, $A \text{ m}^2$, of the stage is given by

$$A = 80x - \left(2 + \frac{\pi}{2}\right)x^2. \quad (4)$$

- (b) Use calculus to find the value of x at which A has a stationary value.

(4)

- (c) Prove that the value of x you found in part (b) gives the maximum value of A .

(2)

- (d) Calculate, to the nearest m^2 , the maximum area of the stage.

(2)

$$a) \quad P = 2x + y + y + \frac{2\pi x}{2}$$

$$P = 2x + 2y + \pi x$$

$$80 = (2 + \pi)x + 2y$$

$$80 - (2 + \pi)x = 2y$$

$$y = 40 - \frac{(2+\pi)x}{2}$$

$$\text{Area} = 2xy + \frac{\pi x^2}{2}$$

$$\text{Area} = 2x \left[40 - \frac{(2+\pi)x}{2} \right] + \frac{\pi x^2}{2}$$

$$= 80x - (2+\pi)x^2 + \frac{\pi x^2}{2}$$

$$= 80x - 2x^2 - \pi x^2 + \frac{\pi x^2}{2}$$

$$= 80x - 2x^2 - \frac{\pi x^2}{2}$$

$$A = 80x - \left(2 + \frac{\pi}{2}\right)x^2$$

$$b) \quad \frac{dA}{dx} = 80 - 2\left(2 + \frac{\pi}{2}\right)x$$

$$\text{At st. pt.} \quad \frac{dA}{dx} = 0$$

$$80 - 2\left(2 + \frac{\pi}{2}\right)x = 0$$

$$80 = 2\left(2 + \frac{\pi}{2}\right)x$$

$$40 = \left(2 + \frac{\pi}{2}\right)x$$

$$\frac{40}{\left(2 + \frac{\pi}{2}\right)} = x$$

$$x = 11.20198$$

$$x = 11.2 \text{ m} \quad \text{to 3 s.f.}$$

$$c) \quad \frac{d^2 A}{dx^2} = -2\left(2 + \frac{\pi}{2}\right) < 0$$

\therefore Maximum at $x = 11.2 \text{ m}$

$$d) \quad A = 80 \times 11.2 - \left(2 + \frac{\pi}{2}\right) \times 11.2^2$$

$$A = 448.079$$

$$A = 448 \text{ m}^2 \quad \text{to nearest m}^2$$

8. A diesel lorry is driven from Birmingham to Bury at a steady speed of v kilometres per hour. The total cost of the journey, £ C , is given by

$$C = \frac{1400}{v} + \frac{2v}{7}.$$

- (a) Find the value of v for which C is a minimum.

(5)

- (b) Find $\frac{d^2C}{dv^2}$ and hence verify that C is a minimum for this value of v .

(2)

- (c) Calculate the minimum total cost of the journey.

(2)

a)

$$C = 1400v^{-1} + \frac{2v}{7}$$

$$\frac{dC}{dv} = -1400v^{-2} + \frac{2}{7}$$

$$\text{Min when } \frac{dC}{dv} = 0 \quad \Rightarrow \quad -\frac{1400}{v^2} + \frac{2}{7} = 0$$

$$\frac{1400}{v^2} = \frac{2}{7}$$

$$9800 = 2v^2$$

$$4900 = v^2$$

$$\sqrt{4900} = v$$

$$70 = v$$

$$v = 70 \text{ km/h}$$

b)

$$\frac{d^2C}{dv^2} = 2800v^{-3}$$



$$\frac{d^2C}{dv^2} = \frac{2800}{v^3}$$

$$\text{when } v=70, \quad \frac{d^2C}{dv^2} = \frac{2800}{70^3} > 0 \quad \therefore \text{minimum}$$

c)

$$C = \frac{1400}{70} + \frac{2 \times 70}{7}$$

$$C = 20 + 20$$

$$C = £40$$

10. The volume $V \text{ cm}^3$ of a box, of height $x \text{ cm}$, is given by

$$V = 4x(5-x)^2, \quad 0 < x < 5$$

(a) Find $\frac{dV}{dx}$. (4)

(b) Hence find the maximum volume of the box. (4)

(c) Use calculus to justify that the volume that you found in part (b) is a maximum. (2)

a) $V = 4x(25 - 10x + x^2)$

$$V = 100x - 40x^2 + 4x^3$$

$$\frac{dV}{dx} = 100 - 80x + 12x^2$$

b) Max when $\frac{dV}{dx} = 0$

$$\Rightarrow 12x^2 - 80x + 100 = 0$$

By calc $x = 5$ or $x = \frac{5}{3}$

$x = 5$, $V = 4(5)(5-5)^2 = 0$

$x = \frac{5}{3}$, $V = 4\left(\frac{5}{3}\right)\left(5 - \frac{5}{3}\right)^2$

$$V = \frac{2000}{27} = 74.1 \text{ cm}^3 \text{ to 3 s.f.}$$

c) $\frac{d^2V}{dx^2} = 24x - 80$, $x = \frac{5}{3}$, $\frac{d^2V}{dx^2} = 24\left(\frac{5}{3}\right) - 80 = -40 < 0$
 \therefore a maximum



8.

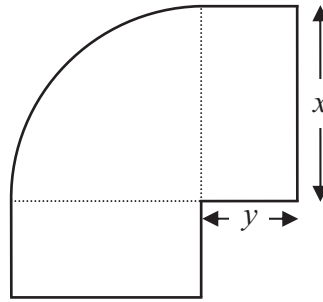


Figure 3

Figure 3 shows a flowerbed. Its shape is a quarter of a circle of radius x metres with two equal rectangles attached to it along its radii. Each rectangle has length equal to x metres and width equal to y metres.

Given that the area of the flowerbed is 4 m^2 ,

(a) show that

$$y = \frac{16 - \pi x^2}{8x} \quad (3)$$

(b) Hence show that the perimeter P metres of the flowerbed is given by the equation

$$P = \frac{8}{x} + 2x \quad (3)$$

(c) Use calculus to find the minimum value of P .

(5)

(d) Find the width of each rectangle when the perimeter is a minimum.
Give your answer to the nearest centimetre.

(2)

$$\begin{aligned}
 \text{a) } \text{Area} &= xy + xy + \frac{\pi x^2}{4} = 4 \text{ m}^2 \\
 2xy &= 4 - \frac{\pi x^2}{4} \\
 y &= \frac{4 - \frac{\pi x^2}{4}}{2x} = \frac{16 - \pi x^2}{8x}
 \end{aligned}$$



$$b) \quad \text{Perimeter} = 4y + 2x + \frac{2\pi x}{4}$$

$$= 4\left(\frac{16 - \pi x^2}{8x}\right) + 2x + \frac{2\pi x}{4}$$

$$= \frac{16 - \pi x^2}{2x} + 2x + \frac{2\pi x}{4}$$

$$= \frac{8}{x} - \cancel{\frac{\pi x}{2}} + 2x + \cancel{\frac{\pi x}{2}}$$

$$\text{Perimeter} = \frac{8}{x} + 2x$$

$$c) \quad P = 8x^{-1} + 2x$$

$$\frac{dP}{dx} = -8x^{-2} + 2$$

$$\text{Min when } \frac{dP}{dx} = 0 \quad \Rightarrow \quad -\frac{8}{x^2} + 2 = 0$$

$$2 = \frac{8}{x^2}$$

$$2x^2 = 8$$

$$x^2 = 4$$

$$\underline{x = 2}$$

$$\text{Min value of } P = \frac{8}{2} + 2(2) = 8 \text{ m}$$

Check this is a minimum

$$\frac{d^2P}{dx^2} = 16x^{-3} = \frac{16}{x^3} > 0 \text{ for } x=2 \therefore \text{min}$$

$$d) \quad y = \frac{16 - \pi x^2}{8x}$$

when $x=2$

$$y = \frac{16 - 4\pi}{16}$$

$$y = 0.2146 \text{ cm}$$

$$y = 21 \text{ cm to nearest cm}$$

9. A solid glass cylinder, which is used in an expensive laser amplifier, has a volume of $75\pi \text{ cm}^3$.

The cost of polishing the surface area of this glass cylinder is £2 per cm^2 for the curved surface area and £3 per cm^2 for the circular top and base areas.

Given that the radius of the cylinder is $r \text{ cm}$,

- (a) show that the cost of the polishing, £ C , is given by

$$C = 6\pi r^2 + \frac{300\pi}{r} \quad (4)$$

- (b) Use calculus to find the minimum cost of the polishing, giving your answer to the nearest pound. (5)

- (c) Justify that the answer that you have obtained in part (b) is a minimum. (1)

a) $V = \pi r^2 h = 75\pi$

$$\Rightarrow h = \frac{75\pi}{\pi r^2} = \frac{75}{r^2}$$

$$\text{Curved surface area} = 2\pi r h$$

$$= 2\pi r \times \frac{75}{r^2}$$

$$= \frac{150\pi}{r}$$

$$\text{Top and Bottom Area} = \pi r^2 + \pi r^2 = 2\pi r^2$$

$$\text{Cost of polishing } C = 2\left(\frac{150\pi}{r}\right) + 3(2\pi r^2)$$

$$C = \frac{300\pi}{r} + 6\pi r^2$$



b)

$$C = 300\pi r^{-1} + 6\pi r^2$$

$$\frac{dC}{dr} = -300\pi r^{-2} + 12\pi r$$

$$\text{Min when } \frac{dC}{dr} = 0 \Rightarrow -\frac{300\pi}{r^2} + 12\pi r = 0$$

$$\Rightarrow \frac{300\pi}{r^2} = 12\pi r$$

$$300\pi = 12\pi r^3$$

$$\frac{300\pi}{12\pi} = r^3$$

$$25 = r^3$$

$$r = \sqrt[3]{25}$$

$$r = 2.924 \text{ cm}$$

$$\text{Min Cost} = \frac{300\pi}{2.924} + 6\pi \times 2.924^2$$

$$= £483.48$$

$$= £483 \quad \text{to nearest } £$$

$$\begin{aligned} c) \quad \frac{d^2C}{dr^2} &= 600\pi r^{-3} + 12\pi \\ &= \frac{600\pi}{r^3} + 12\pi > 0 \quad \text{for all } r > 0 \end{aligned}$$

\therefore a min at $r = 2.924 \text{ cm}$

9.

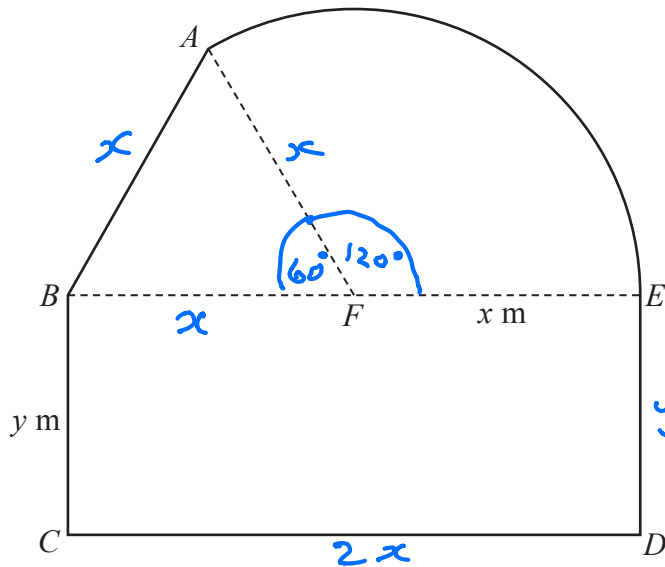


Diagram not drawn to scale

Figure 4

Figure 4 shows a plan view of a sheep enclosure.

The enclosure $ABCDEA$, as shown in Figure 4, consists of a rectangle $BCDE$ joined to an equilateral triangle BFA and a sector FEA of a circle with radius x metres and centre F .

The points B , F and E lie on a straight line with $FE = x$ metres and $10 \leq x \leq 25$

- (a) Find, in m^2 , the exact area of the sector FEA , giving your answer in terms of x , in its simplest form.

(2)

Given that $BC = y$ metres, where $y > 0$, and the area of the enclosure is 1000 m^2 ,

- (b) show that

$$y = \frac{500}{x} - \frac{x}{24}(4\pi + 3\sqrt{3})$$

(3)

- (c) Hence show that the perimeter P metres of the enclosure is given by

$$P = \frac{1000}{x} + \frac{x}{12}(4\pi + 36 - 3\sqrt{3})$$

(3)

- (d) Use calculus to find the minimum value of P , giving your answer to the nearest metre.

(5)

- (e) Justify, by further differentiation, that the value of P you have found is a minimum.

(2)

a) Area of Sector = $\frac{\pi x^2}{3}$



b) whole shape

$$\text{Area} = 2xy + \frac{\pi x^2}{3} + \frac{1}{2}x \cdot x \sin 60^\circ$$

$$1000 = 2xy + \frac{\pi x^2}{3} + \frac{\sqrt{3}}{4}x^2$$

$$1000 - \frac{\pi x^2}{3} - \frac{\sqrt{3}}{4}x^2 = 2xy$$

$$\frac{1000}{2x} - \frac{\pi x^2}{6x} - \frac{\sqrt{3}x^2}{8x} = y$$

$$y = \frac{500}{x} - \frac{\pi x}{6} - \frac{\sqrt{3}x}{8}$$

$$y = \frac{500}{x} - \frac{x}{24}(4\pi - 3\sqrt{3})$$

c) $P = 2x + 2y + \frac{2\pi x}{3} + x$

$$P = 2x + 2\left(\frac{500}{x} - \frac{x}{24}(4\pi + 3\sqrt{3})\right) + \frac{2\pi x}{3} + x$$

$$P = 3x + \frac{2\pi x}{3} + \frac{1000}{x} - \frac{x}{12}(4\pi + 3\sqrt{3})$$

$$P = \frac{1000}{x} + \frac{36x}{12} + \frac{8\pi x}{12} - \frac{4\pi x}{12} - \frac{3\sqrt{3}x}{12}$$

$$P = \frac{1000}{x} + \frac{x}{12}[36 + 4\pi - 3\sqrt{3}]$$

$$d) \quad P = 1000x^{-1} + \frac{x}{12} [36 + 4\pi - 3\sqrt{3}]$$

$$\frac{dP}{dx} = -1000x^{-2} + \frac{(36 + 4\pi - 3\sqrt{3})}{12}$$

$$\frac{dP}{dx} = -\frac{1000}{x^2} + \frac{(36 + 4\pi - 3\sqrt{3})}{12}$$

$$\text{At st pt } \frac{dP}{dx} = 0$$

$$\frac{1000}{x^2} = \frac{36 + 4\pi - 3\sqrt{3}}{12}$$

$$\frac{x^2}{1000} = \frac{12}{36 + 4\pi - 3\sqrt{3}}$$

$$x^2 = \frac{12000}{36 + 4\pi - 3\sqrt{3}}$$

$$x = \sqrt{\frac{12000}{(36 + 4\pi - 3\sqrt{3})}}$$

$$x = 16.63392808$$

$$x = 16.6 \text{ m}$$

$$P = \frac{1000}{16.63392808} + \frac{16.63392808}{12} (4\pi + 36 - 3\sqrt{3})$$

$$P = 120.236 \text{ m}$$

$$\text{Min } P = 120 \text{ m to nearest m}$$

$$e) \quad \frac{d^2 A}{dx^2} = -2(-1000x^{-3}) = \frac{2000}{x^3}$$

$$\text{when } x = 16.6, \quad \frac{d^2 A}{dx^2} = \frac{2000}{16.6^3} > 0$$

\therefore a minimum at $x = 16.6 \text{ m}$
