Vectors in Kinematics and Statics Problems Solutions

1 The position vector, $\mathbf{r}$, of a particle of mass 4 kg at time t is given by

$$\mathbf{r} = t^2\mathbf{i} + (5t - 2t^2)\mathbf{j},$$

where $\mathbf{i}$ and $\mathbf{j}$ are the standard unit vectors, lengths are in metres and time is in seconds.

(i) Find an expression for the acceleration of the particle. [4]

The particle is subject to a force $\mathbf{F}$ and a force $12\mathbf{j}$ N.

(ii) Find $\mathbf{F}$. [3]

i)

$$\underline{r} = \begin{pmatrix} t^2 \\ 5t - 2t^2 \end{pmatrix}$$

$$\underline{v} = \frac{d\underline{r}}{dt} = \begin{pmatrix} 2t \\ 5 - 4t \end{pmatrix}$$

$$\underline{a} = \frac{d\underline{v}}{dt} = \begin{pmatrix} 2 \\ -4 \end{pmatrix}$$

$$\underline{a} = 2\underline{i} - 4\underline{j}$$

ii) N2L $F = ma$

$$\begin{pmatrix} F_1 \\ F_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 12 \end{pmatrix} = 4\begin{pmatrix} 2 \\ -4 \end{pmatrix}$$

$$\begin{pmatrix} F_1 \\ F_2 \end{pmatrix} = \begin{pmatrix} 8 \\ -16 \end{pmatrix} - \begin{pmatrix} 0 \\ 12 \end{pmatrix} = \begin{pmatrix} 8 \\ -28 \end{pmatrix}$$

$$\underline{F} = 8\underline{i} - 28\underline{j} \text{ N}$$

3 A particle rests on a smooth, horizontal plane. Horizontal unit vectors **i** and **j** lie in this plane. The particle is in equilibrium under the action of the three forces $(-3\mathbf{i} + 4\mathbf{j})$ N and $(21\mathbf{i} - 7\mathbf{j})$ N and **R** N.

(i) Write down an expression for **R** in terms of **i** and **j**. [2]

(ii) Find the magnitude of **R** and the angle between **R** and the **i** direction. [4]

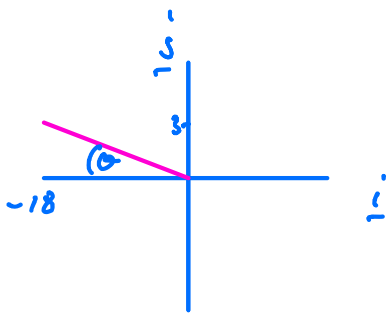
i)
$$\begin{pmatrix} -3 \\ 4 \end{pmatrix} + \begin{pmatrix} 21 \\ -7 \end{pmatrix} + \begin{pmatrix} R_1 \\ R_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} R_1 \\ R_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} - \begin{pmatrix} -3 \\ 4 \end{pmatrix} - \begin{pmatrix} 21 \\ -7 \end{pmatrix}$$

$$\begin{pmatrix} R_1 \\ R_2 \end{pmatrix} = \begin{pmatrix} -18 \\ 3 \end{pmatrix}$$

$$\underline{R} = -18\underline{i} + 3\underline{j} \quad \text{N}$$

ii)
$$|\underline{R}| = \sqrt{(-18)^2 + 3^2} = 18.2 \text{ N} \quad \text{to 3 s.f.}$$



$$\theta = \tan^{-1}\left(\frac{3}{18}\right) = 9.5^\circ$$

Angle between $\underline{R}$ and $\underline{i}$ direction

$= 180 - 9.5^\circ$

$= 170.5^\circ$

5 The position vector of a particle at time t is given by

$$\mathbf{r} = \tfrac{1}{2}t\mathbf{i} + (t^2 - 1)\mathbf{j},$$

referred to an origin O where $\mathbf{i}$ and $\mathbf{j}$ are the standard unit vectors in the directions of the cartesian axes Ox and Oy respectively.

(i) Write down the value of t for which the x-coordinate of the position of the particle is 2. Find the y-coordinate at this time. [2]

(ii) Show that the cartesian equation of the path of the particle is $y = 4x^2 - 1$. [2]

(iii) Find the coordinates of the point where the particle is moving at 45° to both Ox and Oy. [3]

i) $\underline{r} = \begin{pmatrix} \frac{1}{2}t \\ t^2-1 \end{pmatrix}$

$t = 4$ when $x = 2$

$y = 4^2 - 1 = 15$

$y = 15$ when $x = 2$

ii) $x = \frac{1}{2}t$ (1) $\Rightarrow t = 2x$

$y = t^2 - 1$ (2)

Sub for t in (2)

$y = (2x)^2 - 1$

$y = 4x^2 - 1$

iii) $\underline{v} = \dfrac{d\underline{r}}{dt} = \begin{pmatrix} \frac{1}{2} \\ 2t \end{pmatrix}$

when travelling at 45° to Ox and Oy velocity components are equal

$\frac{1}{2} = 2t$

$\frac{1}{4} = t$

when $t = \frac{1}{4}$ $\quad \underline{r} = \begin{pmatrix} \frac{1}{2}(\frac{1}{4}) \\ (\frac{1}{4})^2 - 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{8} \\ -\frac{15}{16} \end{pmatrix}$

$\underline{r} = \frac{1}{8}\underline{i} - \frac{15}{16}\underline{j}$

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3 A force **F** is given by **F** = (3.5**i** + 12**j**) N, where **i** and **j** are horizontal unit vectors east and north respectively.

(i) Calculate the magnitude of **F** and also its direction as a bearing. [3]

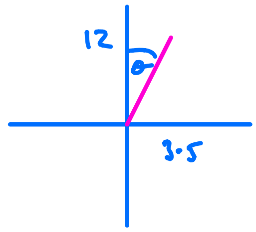
(ii) **G** is the force (7**i** + 24**j**) N. Show that **G** and **F** are in the same direction and compare their magnitudes. [2]

(iii) Force $\mathbf{F}_1$ is (9**i** − 18**j**) N and force $\mathbf{F}_2$ is (12**i** + q**j**) N. Find q so that the sum $\mathbf{F}_1 + \mathbf{F}_2$ is in the direction of **F**. [2]

$\underline{F} = \begin{pmatrix} 3.5 \\ 12 \end{pmatrix}$

i) $|\underline{F}| = \sqrt{3.5^2 + 12^2} = 12.5 \text{ N}$

$\theta = \tan^{-1}\left(\frac{3.5}{12}\right) = 16.3^\circ$



Bearing 016°

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ii) $\underline{G} = \begin{pmatrix} 7 \\ 24 \end{pmatrix} = 2\begin{pmatrix} 3.5 \\ 12 \end{pmatrix} = 2\underline{F}$

$\therefore$ $\underline{G}$ is a multiple of $\underline{F}$ and $\therefore$ in same direction

$|\underline{G}| = 2|\underline{F}|$ so $\underline{G}$ has twice the magnitude of $\underline{F}$

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iii) $\underline{F}_1 = \begin{pmatrix} 9 \\ -18 \end{pmatrix}$ $\underline{F}_2 = \begin{pmatrix} 12 \\ q \end{pmatrix}$

$\underline{F}_1 + \underline{F}_2 = \begin{pmatrix} 21 \\ -18+q \end{pmatrix} = k\begin{pmatrix} 3.5 \\ 12 \end{pmatrix}$

$21 = 3.5k \Rightarrow k = \frac{21}{3.5} \Rightarrow k = 6$

$-18 + q = 6(12)$

$-18 + q = 72$

$q = 72 + 18$

$\underline{q = 90}$

5 The acceleration of a particle of mass 4 kg is given by $\mathbf{a} = (9\mathbf{i} - 4t\mathbf{j})\ \mathrm{m\,s^{-2}}$, where **i** and **j** are unit vectors and t is the time in seconds.

(i) Find the acceleration of the particle when $t = 0$ and also when $t = 3$. [1]

(ii) Calculate the force acting on the particle when $t = 3$. [1]

The particle has velocity $(4\mathbf{i} + 2\mathbf{j})\ \mathrm{m\,s^{-1}}$ when $t = 1$.

(iii) Find an expression for the velocity of the particle at time t. [4]

$$\underline{a} = \begin{pmatrix} 9 \\ -4t \end{pmatrix}$$

i) when $t = 0$ $\quad \underline{a} = \begin{pmatrix} 9 \\ 0 \end{pmatrix} = 9\underline{i}$

when $t = 3$ $\quad \underline{a} = \begin{pmatrix} 9 \\ -12 \end{pmatrix} = 9\underline{i} - 12\underline{j}$

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ii) $\underline{F} = m\underline{a}$ $\quad$ when $t = 3$

$$\underline{F} = 4\begin{pmatrix} 9 \\ -12 \end{pmatrix} = \begin{pmatrix} 36 \\ -48 \end{pmatrix}$$

$$\underline{F} = 36\underline{i} - 48\underline{j}$$

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iii) $$\underline{v} = \int \underline{a}\, dt = \int \begin{pmatrix} 9 \\ -4t \end{pmatrix} dt$$

$$\underline{v} = \begin{pmatrix} 9t \\ -2t^2 \end{pmatrix} + \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$t = 1$, $\underline{v} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$ $\quad \begin{pmatrix} 4 \\ 2 \end{pmatrix} = \begin{pmatrix} 9 \\ -2 \end{pmatrix} + \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$

$$\begin{pmatrix} -5 \\ 4 \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$\underline{v} = \begin{pmatrix} 9t - 5 \\ -2t^2 + 4 \end{pmatrix}$ $\quad \underline{v} = (9t - 5)\underline{i} + (-2t^2 + 4)\underline{j}$

2 Force $\mathbf{F}_1$ is $\begin{pmatrix} -6 \\ 13 \end{pmatrix}$N and force $\mathbf{F}_2$ is $\begin{pmatrix} -3 \\ 5 \end{pmatrix}$N, where $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are vectors east and north respectively.

(i) Calculate the magnitude of $\mathbf{F}_1$, correct to three significant figures. [2]

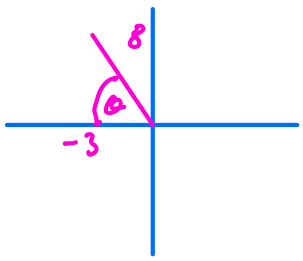
(ii) Calculate the direction of the force $\mathbf{F}_1 - \mathbf{F}_2$ as a bearing. [3]

Force $\mathbf{F}_2$ is the resultant of all the forces acting on an object of mass 5 kg.

(iii) Calculate the acceleration of the object and the change in its velocity after 10 seconds. [3]

i) $|F_1| = \sqrt{(-6)^2 + 13^2} = 14.3 \text{ N}$ to 3 s.f.

ii) $\underline{F_1} - \underline{F_2} = \begin{pmatrix} -6 \\ 13 \end{pmatrix} - \begin{pmatrix} -3 \\ 5 \end{pmatrix} = \begin{pmatrix} -3 \\ 8 \end{pmatrix}$



$\alpha = \tan^{-1}\left(\frac{8}{3}\right) = 69.4°$

$\text{Bearing} = 270 + 69.4°$

$= 339.4°$

$= 339°$ to 3 s.f.

iii) N2L $F = ma$

$\underline{F_2} = \begin{pmatrix} -3 \\ 5 \end{pmatrix} = 5\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$

$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} -0.6 \\ 1 \end{pmatrix}$

$\underline{a} = \begin{pmatrix} -0.6 \\ 1 \end{pmatrix}$

$\underline{v} = \int \underline{a}\, dt = \int \begin{pmatrix} -0.6 \\ 1 \end{pmatrix} dt = \begin{pmatrix} -0.6t + c_1 \\ t + c_2 \end{pmatrix}$

Change in velocity $= \underline{v}_{10} - \underline{v}_0$

$$= \begin{pmatrix} -0.6(10) + c_1 \\ 10 + c_2 \end{pmatrix} - \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$= \begin{pmatrix} -6 \\ 10 \end{pmatrix}$$

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4 Fig. 4 shows the unit vectors **i** and **j** in the directions of the cartesian axes Ox and Oy, respectively. O is the origin of the axes and of position vectors.

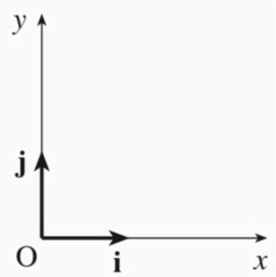

Fig. 4

The position vector of a particle is given by $\mathbf{r} = 3t\mathbf{i} + (18t^2 - 1)\mathbf{j}$ for $t \geqslant 0$, where t is time.

(i) Show that the path of the particle cuts the x-axis just once. [2]

(ii) Find an expression for the velocity of the particle at time t.

Deduce that the particle never travels in the **j** direction. [3]

(iii) Find the cartesian equation of the path of the particle, simplifying your answer. [3]

i) $\underline{r} = \begin{pmatrix} 3t \\ 18t^2 - 1 \end{pmatrix}$

Cuts x-axis when $18t^2 - 1 = 0$

$$18t^2 = 1$$

$$t^2 = \frac{1}{18}$$

$$t = \pm\sqrt{\frac{1}{18}}$$

But $t \geqslant 0$ so $t = +\sqrt{\frac{1}{18}}$ only time it cuts x-axis

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ii) $\underline{v} = \frac{dr}{dt} = \begin{pmatrix} 3 \\ 36t \end{pmatrix}$ $\underline{v} = 3\underline{i} + 36t\underline{j}$

Always has component 3 in $\underline{i}$ direction so never travels in j direction

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iii) $x = 3t \Rightarrow t = \frac{x}{3}$

$y = 18t^2 - 1$

$y = 18\left(\frac{x}{3}\right)^2 - 1$

$y = \frac{18x^2}{9} - 1$

$\underline{y = 2x^2 - 1}$