

- 2 a Use de Moivre's theorem to show that  
 $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$ . (4)

- b Hence find 3 distinct solutions of the equation  $16x^5 - 20x^3 + 5x + 1 = 0$ , giving your answers to 3 decimal places where appropriate. (5)

← Section 1.4

$$\begin{aligned}
 a) \quad & \cos 5\theta + i \sin 5\theta = (\cos\theta + i \sin\theta)^5 \quad \text{de Moivre} \\
 &= (c + is)^5 \\
 &= c^5 + 5c^4is + 10c^3i^2s^2 + 10c^2i^3s^3 + 5ci^4s^4 + i^5s^5 \\
 &= c^5 + 5c^4si - 10c^3s^2 - 10c^2s^3i + 5cs^4 + s^5i
 \end{aligned}$$

Equating Re and Im parts

$$\begin{aligned}
 \cos 5\theta &= c^5 - 10c^3(1-c^2) + 5c(1-c^2)^2 \\
 &= c^5 - 10c^3 + 10c^5 + 5c(1-2c^2+c^4) \\
 &= c^5 - 10c^3 + 10c^5 + 5c - 10c^3 + 5c^5 \\
 &= 16c^5 - 20c^3 + 5c
 \end{aligned}$$

$$\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$$


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$$\begin{aligned}
 b) \quad & 16x^5 - 20x^3 + 5x + 1 = 0 \\
 & 16x^5 - 20x^3 + 5x = -1
 \end{aligned}$$

Let  $x = \cos\theta$

$$16\cos^5\theta - 20\cos^3\theta + 5\cos\theta = -1$$

$$\cos 5\theta = -1$$

$$5\theta = \cos^{-1}(-1) = \pi, 3\pi, 5\pi, 7\pi, 9\pi$$

$$\theta = \frac{\pi}{5}, \frac{3\pi}{5}, \pi, \frac{7\pi}{5}, \frac{9\pi}{5}$$

$$x = \cos \frac{\pi}{5}, \cos \frac{3\pi}{5}, \cos \pi, \cos \frac{7\pi}{5}, \cos \frac{9\pi}{5}$$

$$x = 0.809, -0.309, -1, -0.309, 0.809$$

3 distinct solutions

$$x = 0.809, -0.309, -1$$

first two  
to 3 d.p.

- 7 a Solve the equation

$$z^5 = 4 + 4i$$

giving your answers in the form

$z = re^{ik\pi}$ , where  $r$  is the modulus of  $z$   
and  $k$  is a rational number such that

$$0 \leq k \leq 2. \quad (6)$$

- b Show on an Argand diagram the points representing your solutions. (2)

← Section 1.6

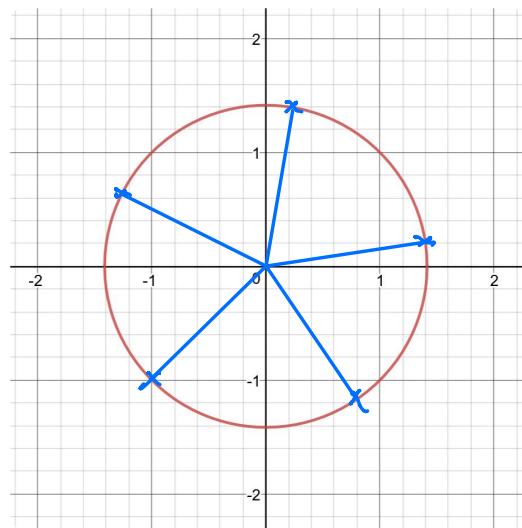
$$z^5 = 4\sqrt{2} e^{i\frac{\pi}{4}} = \sqrt{2}^5 e^{i\frac{\pi}{4}}$$

$$z = \sqrt{2} e^{i\left(\frac{2k\pi}{5} + \frac{\pi}{20}\right)} \quad \text{for } k = 0, 1, 2, 3, 4$$

$$z = \sqrt{2} e^{i\left(\frac{8k\pi + \pi}{20}\right)}$$

$$z = \sqrt{2} e^{i\frac{\pi}{20}}, \sqrt{2} e^{i\frac{9\pi}{20}}, \sqrt{2} e^{i\frac{17\pi}{20}}, \sqrt{2} e^{i\frac{25\pi}{20}}, \sqrt{2} e^{i\frac{33\pi}{20}}$$

b)



12 Prove that

$$\sum_{r=1}^n \frac{2}{(r+1)(r+2)} = \frac{n}{n+2}$$

← Section 2.1

$$(5) \quad \frac{2}{(r+1)(r+2)} = \frac{A}{r+1} + \frac{B}{r+2}$$

$$= \frac{2}{r+1} - \frac{2}{r+2}$$

$$\sum_{r=1}^n \frac{2}{(r+1)(r+2)} = \sum_{r=1}^n \left( \frac{2}{r+1} - \frac{2}{r+2} \right)$$

$$\begin{array}{ccccccc}
 r & \frac{2}{r+1} & - & \frac{2}{r+2} \\
 \hline
 1 & \frac{2}{2} & - & \frac{2}{3} \\
 2 & \frac{2}{3} & - & \frac{2}{4} \\
 3 & \frac{2}{4} & - & \frac{2}{5} \\
 \vdots & & & \\
 n-1 & \frac{2}{n} & - & \frac{2}{n+1} \\
 n & \frac{2}{n+1} & - & \frac{2}{n+2}
 \end{array}$$

$$= \frac{2}{2} - \frac{2}{n+2}$$

$$\begin{aligned}
 &= 1 - \frac{2}{n+2} \\
 &= \frac{n+2-2}{n+2} \\
 &= \frac{n}{n+2}
 \end{aligned}$$


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17 a Prove that

$$\sum_{r=1}^n \frac{4}{r(r+2)} = \frac{n(an+b)}{(n+1)(n+2)}$$

where  $a$  and  $b$  are constants to be found.

(5)

b Find the value of  $\sum_{r=50}^{100} \frac{4}{r(r+2)}$ , to 4 decimal places.

(2)

$$\begin{aligned}
 \frac{4}{r(r+2)} &\equiv \frac{A}{r} + \frac{B}{r+2} \\
 &\equiv \frac{2}{r} - \frac{2}{r+2}
 \end{aligned}$$

← Section 2.1

$$\sum_{r=1}^n \frac{4}{r(r+2)} = \sum_{r=1}^n \left( \frac{2}{r} - \frac{2}{r+2} \right)$$

$$\begin{array}{c}
 \frac{2}{r} - \frac{2}{r+2} \\
 \hline
 r \\
 \hline
 1 \quad \frac{2}{1} - \frac{2}{3} \\
 2 \quad \frac{2}{2} - \frac{2}{4} \\
 3 \quad \frac{2}{3} - \frac{2}{5} \\
 4 \quad \frac{2}{4} - \frac{2}{6} \\
 \vdots \\
 n-2 \quad \frac{2}{n-2} - \frac{2}{n} \\
 n-1 \quad \frac{2}{n-1} - \frac{2}{n+1} \\
 n \quad \frac{2}{n} - \frac{2}{n+2}
 \end{array}$$

$$\begin{aligned}
&= \frac{2}{1} + \frac{2}{2} - \frac{2}{n+1} - \frac{2}{n+2} \\
&= 3 - \frac{2}{n+1} - \frac{2}{n+2} \\
&= \frac{3(n+1)(n+2) - 2(n+2) - 2(n+1)}{(n+1)(n+2)} \\
&= \frac{3(n^2 + 3n + 2) - 2n - 4 - 2n - 2}{(n+1)(n+2)} \\
&= \frac{3n^2 + 9n + 6 - 4n - 6}{(n+1)(n+2)} \\
&= \frac{3n^2 + 5n}{(n+1)(n+2)} \\
&= \frac{n(3n+5)}{(n+1)(n+2)}
\end{aligned}$$


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b)

$$\begin{aligned}
\sum_{r=50}^{100} \frac{4}{r(r+2)} &= \sum_{r=1}^{100} \frac{4}{r(r+2)} - \sum_{r=1}^{49} \frac{4}{r(r+2)} \\
&= \frac{100(305)}{(101)(102)} - \frac{49(152)}{(50)(51)} \\
&= 0.0398 \quad \text{to 4 d.p.}
\end{aligned}$$


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22 Find  $\sum_{r=1}^n \frac{2r+3}{3^r(r+1)}$  (5)

← Section 2.1

Strange question unsuitable for partial fractions

Book gives answer as  $1 - \frac{1}{3^n(n+1)}$

This works for  $n=1$

$$\begin{aligned} \frac{2(1)+3}{3^1(1+1)} &= \frac{5}{6} & 1 - \frac{1}{3^1(1+1)} \\ &= 1 - \frac{1}{6} & = \frac{5}{6} \end{aligned}$$

But not for  $n=2$

$$\begin{aligned} &\frac{5}{6} + \frac{2(2)+3}{3^2(2+1)} & 1 - \frac{1}{3^2(2+1)} \\ &= \frac{5}{6} + \frac{7}{27} &= 1 - \frac{1}{27} \\ &= \frac{59}{54} &= \frac{26}{27} \end{aligned}$$

Not a question to be considered further!

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- 27 a By using the Maclaurin series for  $\cos x$  and  $\ln(1 + x)$ , find the series expansion for  $\ln(\cos x)$  in ascending powers of  $x$  up to and including the term in  $x^4$ . (6)

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} - + \dots$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

$$\begin{aligned}\ln(\cos x) &\approx \ln\left(1 + \left(-\frac{x^2}{2} + \frac{x^4}{24}\right)\right) \\ &\approx \left(-\frac{x^2}{2} + \frac{x^4}{24}\right) - \frac{\left(-\frac{x^2}{2} + \frac{x^4}{24}\right)^2}{2} \\ &\approx -\frac{x^2}{2} + \frac{x^4}{24} - \frac{\left(\frac{x^4}{24}\right)}{2} \\ &\approx -\frac{x^2}{2} - \frac{x^4}{12}\end{aligned}$$


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- b Hence, or otherwise, obtain the first two non-zero terms in the series expansion for  $\ln(\sec x)$  in ascending powers of  $x$ . (4)

← Sections 2.3, 2.4

$$\begin{aligned}\ln(\sec x) &= \ln\left(\frac{1}{\cos x}\right) = \ln 1 - \ln(\cos x) \\ &\approx 0 - \left(-\frac{x^2}{2} - \frac{x^4}{12}\right) \\ &\approx \frac{x^2}{2} + \frac{x^4}{12}\end{aligned}$$


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- 32 Show that  $\int_1^\infty x^3 e^{-x^4} dx$  converges and find its exact value.

(5)

← Section 3.1

$$\int x^3 e^{-x^4} dx = -\frac{1}{4} e^{-x^4} + C \quad \text{by inspection}$$

$$\int_1^\infty x^3 e^{-x^4} dx = \lim_{t \rightarrow \infty} \int_1^t x^3 e^{-x^4} dx$$

$$= \lim_{t \rightarrow \infty} \left[ -\frac{1}{4} e^{-x^4} \right]_1^t$$

$$= \lim_{t \rightarrow \infty} \left[ -\frac{1}{4} e^{-t^4} + \frac{1}{4} e^{-1} \right]$$

$$= 0 + \frac{1}{4e}$$

$$= \frac{1}{4e}$$

37  $f(x) = \frac{1}{\sqrt{3-x}}$

Find the exact mean value of  $f(x)$  on the interval  $[1, 3]$ .

(4)

← Section 3.2

$$\text{Mean Value} = \frac{1}{3-1} \int_1^3 \frac{1}{\sqrt{3-x}} dx = \frac{1}{2} \int_1^3 \frac{1}{\sqrt{3-x}} dx$$

$$x = 3 \Rightarrow u = 0$$

$$x = 1 \Rightarrow u = 2$$

Let  $u = 3 - x$

$$\frac{du}{dx} = -1$$

$$du = -dx$$

$$-du = dx$$

Integral becomes

$$\begin{aligned} \frac{1}{2} \int_2^0 -\frac{1}{\sqrt{u}} du &= \frac{1}{2} \int_0^2 u^{-\frac{1}{2}} du \\ &= \frac{1}{2} \left[ 2u^{\frac{1}{2}} \right]_0^2 \\ &= \frac{1}{2} [2\sqrt{2} - 0] \\ &= \sqrt{2} \end{aligned}$$


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42 Show that

$$\int \frac{2x+1}{x^3+x} dx = A \arctan x + \ln x + B \ln(x^2+1) + C$$

where  $A$  and  $B$  are constants to be found.

(5)

← Section 3.5

$$\frac{2x+1}{x^3+x} = \frac{2x+1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

$$2x+1 \equiv A(x^2+1) + (Bx+C)x$$

$$x=0 \quad 1 = A(0+1) \Rightarrow A = 1$$

$$\text{Coeff of } x^2 \quad 0 = A + B \quad \Rightarrow \quad B = -1$$

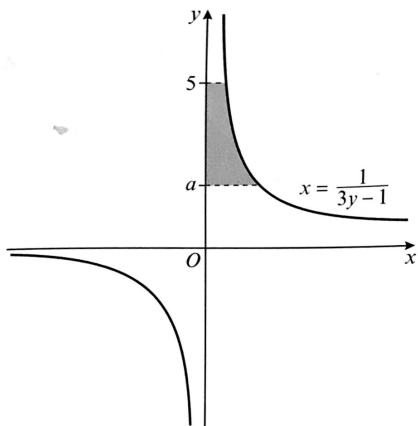
$$\text{Coeff of } x \quad 2 = C \quad \Rightarrow \quad C = 2$$

$$\begin{aligned} \frac{2x+1}{x^3+x} &\equiv \frac{1}{x} + \frac{-x+2}{x^2+1} \\ &\equiv \frac{1}{x} - \frac{x}{x^2+1} + \frac{2}{x^2+1} \end{aligned}$$

$$\begin{aligned} \int \frac{2x+1}{x^3+x} dx &= \int \left( \frac{1}{x} - \frac{x}{x^2+1} + \frac{2}{x^2+1} \right) dx \\ &= \ln x - \frac{1}{2} \ln(x^2+1) + 2 \arctan x + C \end{aligned}$$


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The curve shown has equation  $x = \frac{1}{3y-1}$

The finite region shaded is bounded by the  $y$ -axis, the line  $y = 5$  and the line  $y = a$ .

The region is rotated about the  $y$ -axis through  $360^\circ$ .

Given that the volume of the solid generated is  $\frac{3\pi}{70}$ , find the value of  $a$ . (5)

← Section 4.2

$$Vol = \pi \int_a^5 x^2 dy$$

$$= \pi \int_a^5 \frac{1}{(3y-1)^2} dy$$

$$\text{Let } u = 3y-1$$

$$\frac{du}{dy} = 3$$

$$du = 3dy$$

$$\frac{1}{3} du = dy$$

$$y=5 \quad u=14$$

$$y=a \quad u=3a-1$$

$$\begin{aligned}
 &= \pi \int_{3a-1}^{14} \frac{1}{3u^2} du \\
 &= \pi \left[ -\frac{1}{3u} \right]_{3a-1}^{14} \\
 &= \pi \left[ -\frac{1}{42} + \frac{1}{9a-3} \right]
 \end{aligned}$$

$$\therefore \frac{3\pi}{70} = \pi \left[ -\frac{1}{42} + \frac{1}{9a-3} \right]$$

$$\frac{3}{70} = -\frac{1}{42} + \frac{1}{9a-3}$$

$$\frac{3}{70} + \frac{1}{42} = \frac{1}{9a-3}$$

$$\frac{1}{15} = \frac{1}{9a-3}$$

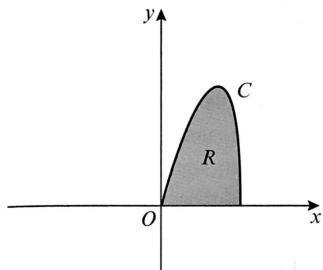
$$\Rightarrow 15 = 9a-3$$

$$15+3 = 9a$$

$$18 = 9a$$

$$\frac{18}{9} = a$$

$$a = 2$$



a)

$$\text{Vol} = \pi \int y^2 dx$$

$$= \pi \int (3 \sin 2t)^2 dx$$

$$x = 2 \sin t$$

$$\frac{dx}{dt} = 2 \cos t$$

$$dx = 2 \cos t dt$$

$$= \pi \int_0^{\frac{\pi}{2}} 9 \sin^2 2t \cdot 2 \cos t dt$$

$$= 18\pi \int_0^{\frac{\pi}{2}} \sin^2 2t \cos t dt$$

The diagram shows the curve  $C$  with parametric equations

$$x = 2 \sin t, y = 3 \sin 2t, 0 \leq t \leq \frac{\pi}{2}$$

A jewellery pendant is made in the shape of the solid of revolution formed when the region marked  $R$  is rotated through  $2\pi$  radians about the  $x$ -axis. Each unit on the axes represents 0.5 cm.

- a Show that the volume of the pendant can be found by evaluating the integral

$$\frac{9\pi}{4} \int_0^{\frac{\pi}{2}} \sin^2 2t \cos t dt \quad (4)$$

- b Hence show that the exact volume of the pendant is  $\frac{6\pi}{5}$  cm<sup>3</sup>. (6)

← Section 4.4

If each unit on axes is  $\frac{1}{2}$  cm

Volume in cm<sup>3</sup> will  
be integral  $\times \left(\frac{1}{2}\right)^3$

$$= \frac{9\pi}{4} \int_0^{\frac{\pi}{2}} \sin^2 2t \cos t dt$$


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$$\begin{aligned} b) \text{ Vol} &= \frac{9\pi}{4} \int_0^{\frac{\pi}{2}} (2 \sin t \cos t)^2 \cos t dt \\ &= 9\pi \int_0^{\frac{\pi}{2}} \sin^2 t (1 - \sin^2 t) \cos t dt \\ &= 9\pi \int_0^{\frac{\pi}{2}} (\sin^2 t - \sin^4 t) \cos t dt \end{aligned}$$

$$\text{Let } u = \sin t$$

$$\frac{du}{dt} = \cos t \quad du = \cos t dt$$

$$t = \frac{\pi}{2} \quad u = 1$$

$$t = 0 \quad u = 0$$

$$\begin{aligned} \text{Vol} &= 9\pi \int_0^1 (u^2 - u^4) du \\ &= 9\pi \left[ \frac{u^3}{3} - \frac{u^5}{5} \right]_0^1 \\ &= 9\pi \left[ \left( \frac{1}{3} - \frac{1}{5} \right) - (0 - 0) \right] \\ &= \frac{6\pi}{5} \text{ cm}^3 \end{aligned}$$

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