

Review Exercise 1 - Q1, Q6, Q11, Q16, Q21, Q26, Q31, Q36, Q41, Q46, Q51

(E) 1 Show that

$$\frac{\cos 2x + i \sin 2x}{\cos 9x - i \sin 9x}$$

can be expressed in the form  
 $\cos nx + i \sin nx$ , where  $n$  is an integer  
 to be found.

(4)  
 ← Section 1.2

(E/P)

$$= \frac{\cos 2x + i \sin 2x}{\cos(-9x) + i \sin(-9x)} \\ = \frac{e^{i2x}}{e^{i(-9x)}} = e^{i11x}$$

$$= \cos 11x + i \sin 11x$$

(E/P) 6 The convergent infinite series  $C$  and  $S$  are defined as

$$C = 1 + \cos \theta + \cos 2\theta + \dots + \cos(n-1)\theta$$

$$S = \sin \theta + \sin 2\theta + \dots + \sin(n-1)\theta$$

By considering  $C + iS$ , show that

$$C = \frac{1 - \cos \theta + \cos(n-1)\theta - \cos n\theta}{2 - 2 \cos \theta}$$

and write down the corresponding expression for  $S$ .

(4)  
 ← Section 1.5

$$C + iS = 1 + (\cos \theta + i \sin \theta) \\ + (\cos 2\theta + i \sin 2\theta) \\ + \dots + (\cos(n-1)\theta + i \sin(n-1)\theta) \\ = 1 + e^{i\theta} + e^{i2\theta} + \dots + e^{i(n-1)\theta}$$

$$\begin{aligned} GP \quad a &= 1 \\ r &= e^{i\theta} \\ n &= n \end{aligned}$$

$$S_n = \frac{a(1 - r^n)}{1 - r}$$

$$\begin{aligned}
 C + iS &= \frac{1(1 - e^{in\theta})}{1 - e^{i\theta}} \\
 &= \frac{(1 - e^{in\theta})}{(1 - e^{i\theta})} \times \frac{(1 - e^{-i\theta})}{(1 - e^{-i\theta})} \\
 &= \frac{1 - e^{in\theta} - e^{-i\theta} + e^{i(n-1)\theta}}{1 - e^{i\theta} - e^{-i\theta} + 1}
 \end{aligned}$$

$$\approx \frac{1 - (\cos n\theta + i \sin n\theta) - (\cos \theta - i \sin \theta) + (\cos(n-1)\theta + i \sin(n-1)\theta)}{2 - (e^{i\theta} + e^{-i\theta})}$$

Equate Re and Im parts

$$C = \frac{1 - \cos n\theta - \cos \theta + \cos(n-1)\theta}{2 - 2 \cos \theta}$$

$$S = \frac{-\sin n\theta + \sin \theta + \sin(n-1)\theta}{2 - 2 \cos \theta}$$

- (E/P) 11 a Write down the five distinct solutions to  $z^5 = 1$ , giving your answers in exponential form, and show that their sum is 0. (4)

- b The point (3, 0) lies at one vertex of a regular pentagon. Given that the pentagon has its centre at the point (2, 1), find the coordinates of the other vertices. (4)

← Section 1.8

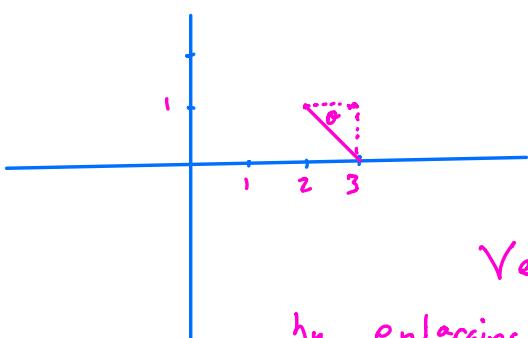
$$Z = e^{i\left(\frac{2k\pi}{5}\right)} \quad \text{for } k \\ = 0, 1, 2, 3, 4$$

$$Z = e^0, e^{i\frac{2\pi}{5}}, e^{i\frac{4\pi}{5}}, \\ e^{i\frac{6\pi}{5}}, e^{i\frac{8\pi}{5}}$$

$$Z = e^0, e^{i\frac{2\pi}{5}}, e^{i\frac{4\pi}{5}}, \\ e^{-i\frac{4\pi}{5}}, e^{-i\frac{2\pi}{5}}$$

$$\begin{aligned} \text{Sum is } & e^0 + e^{i\frac{2\pi}{5}} + e^{-i\frac{2\pi}{5}} + e^{i\frac{4\pi}{5}} + e^{-i\frac{4\pi}{5}} \\ \text{GP } n=5, a = & e^{-i\frac{2\pi}{5}}, r = e^{i\frac{2\pi}{5}} \\ S_5 = & \frac{e^{-i\frac{2\pi}{5}} \left(1 - e^{i\frac{2\pi}{5} \cdot 5}\right)}{1 - e^{i\frac{2\pi}{5}}} = \frac{e^{-i\frac{2\pi}{5}} \left(1 - e^{i2\pi}\right)}{1 - e^{i\frac{2\pi}{5}}} \\ & = \frac{e^{-i\frac{2\pi}{5}} (1 - 1)}{1 - e^{i\frac{2\pi}{5}}} = 0 \end{aligned}$$

Enlargement and translation of answers to



$$\theta = \frac{\pi}{4}$$

Distance from centre  
to a vertex =  $\sqrt{2}$

Vertices will be obtained  
by enlarging previous origin centred pentagon  
by scale factor  $\sqrt{2}$ , rotating it  $\frac{\pi}{4}$  clockwise

and then translating by  $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$$\begin{pmatrix} 2 \\ 1 \end{pmatrix} + \sqrt{2} \begin{pmatrix} \cos\left(\frac{2k\pi}{5} - \frac{\pi}{4}\right) \\ \sin\left(\frac{2k\pi}{5} - \frac{\pi}{4}\right) \end{pmatrix} \quad k = 0, 1, 2, 3, 4$$

$$(3, 0) \quad (3.26, 1.64) \quad (1.78, 2.40)$$

$$(0.60, 1.22) \quad (1.36, -0.26)$$


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- 16 a Express as a simplified single fraction

$$\frac{1}{(r-1)^2} - \frac{1}{r^2} \quad (2)$$

- b Hence prove, by the method of differences, that

$$\sum_{r=2}^n \frac{2r-1}{r^2(r-1)^2} = 1 - \frac{1}{n^2} \quad (3)$$

← Section 2.1

$$= \frac{r^2 - (r-1)^2}{(r-1)^2 r^2}$$

$$= \frac{r^2 - (r^2 - 2r + 1)}{(r-1)^2 r^2}$$

$$= \frac{2r-1}{(r-1)^2 r^2}$$

$$b) \quad \sum_{r=2}^n \frac{2r-1}{(r-1)^2 r^2} = \sum_{r=2}^n \frac{1}{(r-1)^2} - \frac{1}{r^2}$$

$$\frac{r}{(r-1)^2} - \frac{1}{r^2}$$

$$\frac{2}{1} - \frac{1}{2^2}$$

$$\frac{3}{2^2} - \frac{1}{3^2}$$

$$\frac{4}{3^2} - \frac{1}{4^2}$$

$$\vdots$$

$$\begin{aligned}
 & \vdots \\
 n-1 & \frac{1}{(n-2)^2} - \frac{1}{(n-1)^2} \\
 n & \frac{1}{(n-1)^2} - \frac{1}{n^2} \\
 = & 1 - \frac{1}{n^2}
 \end{aligned}$$


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(E) 21 a Show that

$$\frac{r^3 - r + 1}{r(r+1)} = r - 1 + \frac{1}{r} - \frac{1}{r+1} \quad (2)$$

for  $r \neq 0, -1$ .

$$\begin{aligned}
 & r - 1 + \frac{1}{r} - \frac{1}{r+1} \\
 = & \frac{r(r)(r+1) - 1(r)(r+1) + 1(r+1) - 1(r)}{r(r+1)} \\
 = & \frac{r^3 + r^2 - r^2 - r + r + 1 - r}{r(r+1)} \\
 = & \frac{r^3 - r + 1}{r(r+1)}
 \end{aligned}$$

b Find  $\sum_{r=1}^n \frac{r^3 - r + 1}{r(r+1)}$ , expressing your answer as a single fraction in its simplest form. (3)

← Section 2.1

$$\sum_{r=1}^n \frac{r^3 - r + 1}{r(r+1)} = \sum_{r=1}^n \left( r - 1 + \frac{1}{r} - \frac{1}{r+1} \right)$$

$$\sum_{r=1}^n (r-1) = \sum_{r=1}^n r - \sum_{r=1}^n 1 = \frac{1}{2}n(n+1) - n$$

$$\begin{array}{c} \sum_{r=1}^n \left( \frac{1}{r} - \frac{1}{r+1} \right) \\ \hline r & \frac{1}{r} & - \frac{1}{r+1} \\ \hline 1 & 1 & - \frac{1}{2} \\ 2 & \frac{1}{2} & - \frac{1}{3} \\ 3 & \frac{1}{3} & - \frac{1}{4} \\ \vdots & \vdots & \vdots \\ n-1 & \frac{1}{n-1} & - \frac{1}{n} \\ n & \frac{1}{n} & - \frac{1}{n+1} \end{array}$$

$$= 1 - \frac{1}{n+1}$$

$$\begin{aligned} \therefore \sum_{r=1}^n \frac{r^3 - r + 1}{r(r+1)} &= \frac{1}{2}n(n+1) - n + 1 - \frac{1}{n+1} \\ &= \frac{n^2}{2} + \frac{n}{2} - n + 1 - \frac{1}{n+1} \\ &= \frac{n^2}{2} - \frac{n}{2} + 1 - \frac{1}{n+1} \\ &= \frac{n^2(n+1) - n(n+1) + 2(n+1) - 2}{2(n+1)} \\ &= \frac{\cancel{n^3} + \cancel{n^2} - \cancel{n^2} - n + 2n + \cancel{2} - \cancel{2}}{2(n+1)} \\ &= \frac{n^3 + n}{2(n+1)} \\ &= \frac{n(n^2 + 1)}{2(n+1)} \end{aligned}$$

- 26 a Find the first four terms of the expansion, in ascending powers of  $x$ , of  
 $(2x+3)^{-1}, |x| < \frac{2}{3}$  (3)

- b Hence, or otherwise, find the first four non-zero terms of the expansion, in ascending powers of  $x$ , of

$$\frac{\sin 2x}{2x+3}, |x| < \frac{2}{3} \quad (5)$$

← Sections 2.3, 2.4

$$\frac{1}{2x+3} = \frac{1}{3\left(1 + \frac{2x}{3}\right)}$$

$$= \frac{1}{3} \left(1 + \frac{2x}{3}\right)^{-1}$$

$$= \frac{1}{3} \left[ 1 + -1\left(\frac{2x}{3}\right) + \frac{-1 \cdot -2}{1 \cdot 2} \left(\frac{2x}{3}\right)^2 + \frac{-1 \cdot -2 \cdot -3}{1 \cdot 2 \cdot 3} \left(\frac{2x}{3}\right)^3 + \dots \right]$$

$$= \frac{1}{3} \left[ 1 - \frac{2x}{3} + \frac{4x^2}{9} - \frac{8x^3}{27} \dots \right]$$

$$= \frac{1}{3} - \frac{2x}{9} + \frac{4x^2}{27} - \frac{8x^3}{81} + \dots$$


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b)  $\sin 2x = 2x - \frac{(2x)^3}{3!} + \dots = 2x - \frac{4x^3}{3} + \dots$

$$\frac{\sin 2x}{2x+3} \approx \left(2x - \frac{4x^3}{3}\right) \left(\frac{1}{3} - \frac{2x}{9} + \frac{4x^2}{27} - \frac{8x^3}{81}\right)$$

$$\approx \frac{2x}{3} - \frac{4x^2}{9} + \frac{8x^3}{27} - \frac{16x^4}{81} \\ - \frac{4x^3}{9} + \frac{8x^4}{27}$$

$$\approx \frac{2x}{3} - \frac{4x^2}{9} - \frac{4x^3}{27} + \frac{8x^4}{81}$$


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④ 31 a Find  $\int \frac{1}{x(x+3)} dx$

$$(3) \quad \frac{1}{x(x+3)} = \frac{\frac{1}{3}}{x} + \frac{-\frac{1}{3}}{x+3}$$

b Hence show that  $\int_3^\infty \frac{1}{x(x+3)} dx$

converges and find its value.

(3)  
← Section 3.1

$$= \frac{1}{3x} - \frac{1}{3(x+3)}$$

$$\begin{aligned} \int \frac{1}{x(x+3)} dx &= \int \left( \frac{1}{3x} - \frac{1}{3(x+3)} \right) dx \\ &= \frac{1}{3} \int \left( \frac{1}{x} - \frac{1}{x+3} \right) dx \\ &= \frac{1}{3} \ln \left( \frac{x}{x+3} \right) + c \end{aligned}$$


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b)

$$\begin{aligned} \int_3^\infty \frac{1}{x(x+3)} dx &\stackrel{\text{lim } t \rightarrow \infty}{=} \int_3^t \frac{1}{x(x+3)} dx \\ &= \lim_{t \rightarrow \infty} \frac{1}{3} \left[ \ln \left( \frac{x}{x+3} \right) \right]_3^t \\ &= \lim_{t \rightarrow \infty} \frac{1}{3} \left[ \ln \left( \frac{t}{t+3} \right) - \ln \left( \frac{3}{6} \right) \right] \end{aligned}$$

As  $t \rightarrow \infty$

$$\frac{t}{t+3} \rightarrow 1$$

$$\begin{aligned} &= \frac{1}{3} \left[ \ln 1 - \ln \frac{1}{2} \right] \\ &= \frac{1}{3} \left[ 0 - \ln \frac{1}{2} \right] \\ &= \frac{1}{3} \left[ -\ln \frac{1}{2} \right] \\ &= \frac{1}{3} \ln 2 \end{aligned}$$


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(E/P) 36  $f(x) = x^2(x^3 - 1)^3$

a Show that the mean value of  $f(x)$  over the interval  $[1, 3]$  is  $\frac{57122}{3}$  (3)

b Use the answer to part a to find the mean value over the interval  $[1, 3]$  of  $-2f(x)$ . (2)

← Section 3.2

$$\begin{aligned} & \int f(x) dx \\ &= \int \frac{1}{3} u^3 du \\ &= \frac{u^4}{12} + C \\ &= \frac{(x^3 - 1)^4}{12} + C \end{aligned}$$

Let  $u = x^3 - 1$

$$\frac{du}{dx} = 3x^2$$

$$du = 3x^2 dx$$

$$\frac{1}{3} du = x^2 dx$$

Mean Value of  $f(x)$  over  $[1, 3]$

$$\begin{aligned} &= \frac{1}{3-1} \int_1^3 f(x) dx \\ &= \frac{1}{2} \left[ \frac{(x^3 - 1)^4}{12} \right]_1^3 \\ &= \frac{1}{24} \left[ (3^3 - 1)^4 - (1^3 - 1)^4 \right] \\ &= \frac{57122}{3} \end{aligned}$$

b) Mean Value of  $-2f(x) = -2 \times \frac{57122}{3} = -\frac{114244}{3}$

41  $f(x) = \arcsin x$

a) Show that  $f'(x) = \frac{1}{\sqrt{1-x^2}}$  (3)

b) Given that  $y = \arcsin 2x$ , obtain  $\frac{dy}{dx}$  as an algebraic fraction. (3)

c) Using the substitution  $x = \frac{1}{2}\sin \theta$ , show that

$$\int_0^{\frac{1}{2}} \frac{x \arcsin 2x}{\sqrt{1-4x^2}} dx = \frac{1}{48}(6 - \pi\sqrt{3}). \quad (4)$$

$\leftarrow$  Sections 3.3, 3.4

a) Let  $y = \arcsin x$

$$\sin y = x$$

$$\cos y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\cos y}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-\sin^2 y}}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

b)  $y = \arcsin 2x$

Let  $u = 2x$

$y = \arcsin u$

$$\frac{du}{dx} = 2$$

$$\frac{dy}{du} = \frac{1}{\sqrt{1-u^2}}$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{\sqrt{1-u^2}} \times 2$$

$$= \frac{2}{\sqrt{1-4x^2}}$$

c)

$$\int_0^{\frac{1}{2}} \frac{x \arcsin 2x}{\sqrt{1-4x^2}} dx$$

Let  $x = \frac{1}{2}\sin \theta$

$$\frac{dx}{d\theta} = \frac{1}{2}\cos \theta$$

$$dx = \frac{1}{2}\cos \theta d\theta$$

$$\sin \theta = 2x \\ \Rightarrow \cos \theta = \sqrt{1 - \sin^2 \theta} = \sqrt{1 - 4x^2} \quad \therefore \frac{2 dx}{\cos \theta} = d\theta$$

$$\frac{2 dx}{\sqrt{1 - 4x^2}} = d\theta$$

$$\text{Also } 2x = \sin \theta \\ \Rightarrow \arcsin(2x) = \theta \\ \frac{dx}{\sqrt{1 - 4x^2}} = \frac{1}{2} d\theta$$

When  $x = \frac{1}{4}$

$$\frac{1}{4} = \frac{1}{2} \sin \theta$$

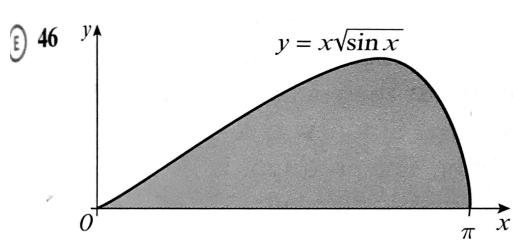
$$\frac{1}{2} = \sin \theta$$

$$\theta = \frac{\pi}{6}$$

Integral becomes

$$\int_0^{\frac{\pi}{6}} \frac{1}{2} \sin \theta \cdot \theta \cdot \frac{1}{2} d\theta \\ = \frac{1}{4} \int_0^{\frac{\pi}{6}} \theta \sin \theta d\theta \\ \text{Let } u = \theta \quad \text{Let } \frac{dv}{d\theta} = \sin \theta \\ \frac{du}{d\theta} = 1 \quad v = -\cos \theta \\ \int u \frac{dv}{d\theta} = uv - \int v \frac{du}{d\theta}$$

$$= \frac{1}{4} \left[ -\theta \cos \theta + \int \cos \theta d\theta \right]_0^{\frac{\pi}{6}} \\ = \frac{1}{4} \left[ -\theta \cos \theta + \sin \theta \right]_0^{\frac{\pi}{6}} \\ = \frac{1}{4} \left[ \left( -\frac{\pi}{6} \cos \frac{\pi}{6} + \sin \frac{\pi}{6} \right) - (0 + 0) \right] = \frac{1}{4} \left[ -\frac{\sqrt{3}\pi}{12} + \frac{1}{2} \right] = \frac{6 - \pi\sqrt{3}}{48}$$



The figure shows a graph of  $y = x\sqrt{\sin x}$ ,  $0 < x < \pi$ .

The finite region enclosed by the curve and the  $x$ -axis is shaded as shown in the figure. A solid body  $S$  is generated by rotating this region through  $2\pi$  radians about the  $x$ -axis. Find the exact volume of  $S$ . (8)

← Section 4.1

$$\text{Vol} = \pi \int_0^\pi y^2 dx$$

$$= \pi \int_0^\pi x^2 \sin x dx$$

Repeated use of integration by parts

$$\int x^2 \sin x dx$$

$$\begin{aligned} \text{Let } u &= x^2 & \text{Let } \frac{dv}{dx} = \sin x \\ \Rightarrow \frac{du}{dx} &= 2x & \Rightarrow v = -\cos x \end{aligned}$$

$$\int x^2 \sin x dx = -x^2 \cos x + \int 2x \cos x dx$$

$$\int 2x \cos x dx$$

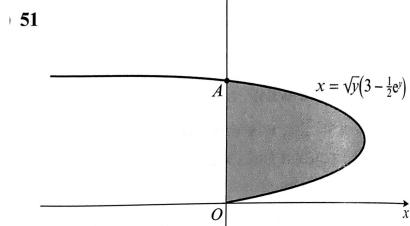
$$\begin{aligned} \text{Let } u &= 2x & \text{Let } \frac{dv}{dx} = \cos x \\ \frac{du}{dx} &= 2 & v = \sin x \end{aligned}$$

$$\begin{aligned} \int 2x \cos x dx &= 2x \sin x - \int 2 \sin x dx \\ &= 2x \sin x + 2 \cos x + C \end{aligned}$$

$$\therefore \int x^2 \sin x dx = -x^2 \cos x + 2x \sin x + 2 \cos x + C$$

$$\pi \int_0^\pi x^2 \sin x dx = \pi \left[ -x^2 \cos x + 2x \sin x + 2 \cos x \right]_0^\pi$$

$$\begin{aligned}
 &= \pi \left[ (-\pi^2 \cos \pi + 2\pi \sin \pi + 2\cos \pi) - (0 + 0 + 2) \right] \\
 &= \pi \left[ \pi^2 + 0 - 2 - 2 \right] \\
 &= \pi^3 - 4\pi
 \end{aligned}$$



The diagram shows the curve with equation

$$x = \sqrt{y} \left( 3 - \frac{1}{2} e^y \right)$$

- a Show that the coordinates of the point marked  $A$  where the curve crosses the  $y$ -axis are  $(0, \ln 6)$ . (2)

The solid of revolution formed when the shaded region is rotated through  $360^\circ$  about the  $y$ -axis is used to model a prototype of a new type of orthopaedic cushion. The prototype is 3D printed using plastic filament.

Given that each unit on the axes is 1 cm,

- b find, correct to 3 significant figures, the volume of the prototype. (6)

- c Suggest a reason why the amount of filament used to print to model may exceed your answer to part b. (1)

← Section 4.4

a)  $x = \sqrt{y} \left( 3 - \frac{1}{2} e^y \right)$

On  $y$ -axis  $x = 0$

$$0 = \sqrt{y} \left( 3 - \frac{1}{2} e^y \right)$$

$$\Rightarrow y = 0 \text{ or } 3 - \frac{1}{2} e^y = 0$$

$$3 = \frac{1}{2} e^y$$

$$6 = e^y$$

$$\ln 6 = y$$

$$\therefore A(0, \ln 6)$$

b)  $Vol = \pi \int_0^{\ln 6} x^2 dy$

$$= \pi \int_0^{\ln 6} y \left( 3 - \frac{1}{2} e^y \right)^2 dy$$

$$= \pi \int_0^{\ln 6} y \left( 9 - 3e^y + \frac{1}{4} e^{2y} \right) dy$$

$$\text{Let } u = y$$

$$\Rightarrow \frac{du}{dy} = 1$$

$$\text{Let } \frac{dv}{dx} = 9 - 3e^y + \frac{1}{4} e^{2y}$$

$$\Rightarrow v = 9y - 3e^y + \frac{1}{8} e^{2y}$$

$$\int u \frac{dv}{dx} = uv - \int v \frac{du}{dx}$$

$$\begin{aligned}
& \int y(9 - 3e^y + \frac{1}{8}e^{2y}) dy \\
&= y(9y - 3e^y + \frac{1}{8}e^{2y}) - \int (9y - 3e^y + \frac{1}{8}e^{2y}) dy \\
&= y(9y - 3e^y + \frac{1}{8}e^{2y}) - \left( \frac{9y^2}{2} - 3e^y + \frac{1}{16}e^{2y} \right) + C \\
&\pi \int_0^{\ln 6} y(9 - 3e^y + \frac{1}{8}e^{2y}) dy \\
&= \pi \left[ y(9y - 3e^y + \frac{1}{8}e^{2y}) - \left( \frac{9y^2}{2} - 3e^y + \frac{1}{16}e^{2y} \right) \right]_0^{\ln 6} \\
&= \pi \left[ \ln 6 \left( 9\ln 6 - 3e^{\ln 6} + \frac{1}{8}e^{2\ln 6} \right) - \left( \frac{9(\ln 6)^2}{2} - 3e^{\ln 6} + \frac{1}{16}e^{2\ln 6} \right) \right] \\
&\quad - \pi \left[ 0 - (0 - 3 + \frac{1}{16}) \right] \\
&= \pi \left[ 9(\ln 6)^2 - 18\ln 6 + \frac{36}{8}\ln 6 - \frac{9(\ln 6)^2}{2} + 18 - \frac{36}{16} \right] \\
&\quad - \pi \left[ \frac{47}{16} \right] \\
&= \pi \left[ \frac{9(\ln 6)^2}{2} - \frac{27}{2}\ln 6 + \frac{63}{4} - \frac{47}{16} \right] \\
&= 9.65 \text{ cm}^3
\end{aligned}$$

c) Filament may be wasted.