Improper Integrals Exercise 3A

1 c) $\int_0^{\infty} e^{-3x} dx = \lim_{t\to\infty} \int_0^t e^{-3x} dx$

$= \lim_{t\to\infty} \left[-\frac{1}{3} e^{-3x} \right]_0^t$

$= \lim_{t\to\infty} -\frac{1}{3} \left[e^{-3t} - e^{0} \right]$

$= \lim_{t\to\infty} -\frac{1}{3} \left[\frac{1}{e^{3t}} - 1 \right]$

$= -\frac{1}{3} \left[0 - 1 \right]$ since $\frac{1}{e^{3t}} \to 0$ as $t \to \infty$

$= \frac{1}{3}$

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2 c) $\int_0^{\infty} \frac{8x}{\sqrt{1+x^2}} dx = \lim_{t\to\infty} \int_0^t \frac{8x}{\sqrt{1+x^2}} dx$

Let $u = 1+x^2$

$\frac{du}{dx} = 2x$

$du = 2x\,dx$

$= \lim_{t\to\infty} \int_1^{1+t^2} \frac{4}{u^{\frac{1}{2}}} du$

$= \lim_{t\to\infty} \int_1^{1+t^2} 4u^{-\frac{1}{2}} du$

$= \lim_{t\to\infty} \left[8u^{\frac{1}{2}} \right]_1^{1+t^2}$

$$= \lim_{t\to\infty} \left(8\sqrt{1+t^2} - 8 \right)$$

$\to \infty$ as $t \to \infty$

since $\sqrt{1+t^2} \to \infty$ as $t \to \infty$

$\therefore$ Integral diverges

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3 c) $$\int_0^{\ln 3} \frac{e^x}{\sqrt{e^x - 1}}\,dx = \lim_{t\to 0} \int_t^{\ln 3} \frac{e^x}{\sqrt{e^x-1}}\,dx$$

Let $u = e^x - 1$

$$\frac{du}{dx} = e^x$$

$$du = e^x\,dx$$

$$= \lim_{t\to 0} \int_{e^t - 1}^{2} \frac{1}{u^{1/2}}\,du$$

$$= \lim_{t\to 0} \left[2\sqrt{u} \right]_{e^t-1}^{2}$$

$$\lim_{t\to 0} \left(2\sqrt{2} - 2\sqrt{e^t - 1} \right)$$

$$= \left(2\sqrt{2} - 2\sqrt{1-1} \right)$$

$$= 2\sqrt{2} - 0$$

$$= 2\sqrt{2}$$

4c) $\displaystyle\int_0^{\pi} \tan x \, dx = \lim_{t_1 \to \frac{\pi}{2}} \int_0^{t_1} \tan x \, dx + \lim_{t_2 \to \frac{\pi}{2}} \int_{t_2}^{\pi} \tan x \, dx$

Consider $\displaystyle\lim_{t_1 \to \frac{\pi}{2}} \int_0^{t_1} \tan x \, dx$

$$= \lim_{t_1 \to \frac{\pi}{2}} \Big[-\ln \cos x \Big]_0^{t_1}$$

$$= \lim_{t_1 \to \frac{\pi}{2}} \Big(-\ln \cos t_1 - -\ln \cos 0 \Big)$$

$$+ 0$$

diverges since $-\ln \cos t_1 \to -\infty$ as $t_1 \to \frac{\pi}{2}$

Do Q1, Q2, Q3, Q4 parts a and b
