

Edexcel FP3

## Hyperbolic Functions

1. Solve the equation

$$7 \operatorname{sech} x - \tanh x = 5$$

Give your answers in the form  $\ln a$  where  $a$  is a rational number.

(5)

$$\frac{7}{\cosh x} - \frac{\sinh x}{\cosh x} = 5$$

$$7 - \sinh x = 5 \cosh x$$

$$7 - \frac{1}{2}(e^x - e^{-x}) = \frac{5}{2}(e^x + e^{-x})$$

$$14 - e^x + e^{-x} = 5e^x + 5e^{-x}$$

$$0 = 5e^x + 5e^{-x} + e^x - e^{-x} - 14$$

$$0 = 6e^x + 4e^{-x} - 14$$

$$0 = 3e^x + 2e^{-x} - 7$$

$$0 = 3e^{2x} - 7e^x + 2$$

$$0 = (3e^x - 1)(e^x - 2)$$

$$3e^x = 1 \quad e^x = 2$$

$$e^x = \frac{1}{3} \quad x = \ln 2$$

$$x = \ln(\frac{1}{3})$$

$$x = \ln(\frac{1}{3}) \quad \text{or} \quad x = \ln 2$$



Leave  
blank

3. (a) Starting from the definitions of  $\sinh x$  and  $\cosh x$  in terms of exponentials, prove that

$$\cosh 2x = 1 + 2 \sinh^2 x \quad (3)$$

- (b) Solve the equation

$$\cosh 2x - 3 \sinh x = 15,$$

giving your answers as exact logarithms.

a) 
$$\begin{aligned} & 1 + 2 \sinh^2 x \\ &= 1 + 2 \left[ \frac{1}{2}(e^{2x} - e^{-2x}) \right]^2 \\ &= 1 + 2 \left[ \frac{1}{4}(e^{2x} - 2 + e^{-2x}) \right] \\ &= 1 + \frac{1}{2}(e^{2x} - 2 + e^{-2x}) \\ &= 1 - 1 + \frac{1}{2}(e^{2x} + e^{-2x}) \\ &= \cosh 2x \end{aligned} \quad (5)$$

b) 
$$\cosh 2x - 3 \sinh x = 15$$

$$1 + 2 \sinh^2 x - 3 \sinh x = 15$$

$$2 \sinh^2 x - 3 \sinh x - 14 = 0$$

$$(2 \sinh x - 7)(\sinh x + 2) = 0$$

$$\sinh x = \frac{7}{2} \quad \text{or} \quad \sinh x = -2$$

$$x = \operatorname{arsinh}\left(\frac{7}{2}\right) \quad \text{or} \quad x = \operatorname{arsinh}(-2)$$

$$x = \ln\left(\frac{7}{2} + \sqrt{\frac{49}{4} + 1}\right) \quad \text{or} \quad x = \ln(-2 + \sqrt{5})$$

$$x = \ln\left(\frac{7 + \sqrt{53}}{2}\right)$$

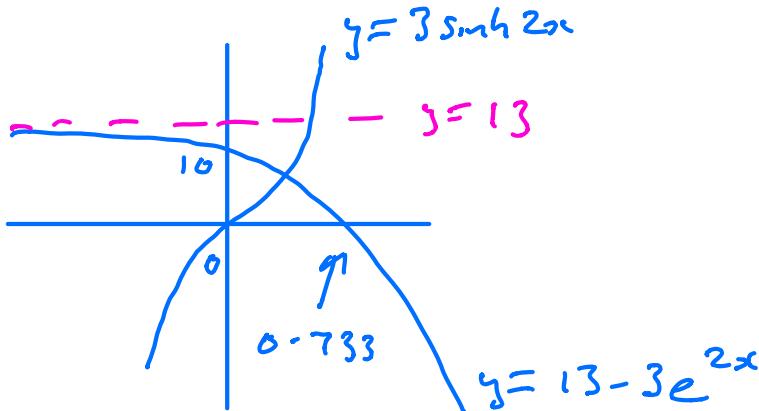


5. The curve  $C_1$  has equation  $y = 3 \sinh 2x$ , and the curve  $C_2$  has equation  $y = 13 - 3e^{2x}$ .

(a) Sketch the graph of the curves  $C_1$  and  $C_2$  on one set of axes, giving the equation of any asymptote and the coordinates of points where the curves cross the axes. (4)

(b) Solve the equation  $3 \sinh 2x = 13 - 3e^{2x}$ , giving your answer in the form  $\frac{1}{2} \ln k$ , where  $k$  is an integer. (5)

a)



$$\begin{aligned} 13 - 3e^{2x} &= 0 \\ e^{2x} &= \frac{13}{3} \\ x &= \frac{1}{2} \ln\left(\frac{13}{3}\right) \\ &= 0.733 \end{aligned}$$

b)

$$3 \sinh 2x = 13 - 3e^{2x}$$

$$\frac{3}{2} (e^{2x} - e^{-2x}) = 13 - 3e^{2x}$$

$$3e^{2x} - 3e^{-2x} = 26 - 6e^{2x}$$

$$9e^{2x} - 3e^{-2x} - 26 = 0$$

$$9e^{4x} - 26e^{2x} - 3 = 0$$

$$(9e^{2x} + 1)(e^{2x} - 3) = 0$$

$$9e^{2x} + 1 = 0$$

no roots

$$e^{2x} - 3 = 0$$

$$e^{2x} = 3$$

$$2x = \ln 3$$

$$x = \frac{1}{2} \ln 3$$



Leave  
blank

7.

$$f(x) = 5 \cosh x - 4 \sinh x, \quad x \in \mathbb{R}$$

(a) Show that  $f(x) = \frac{1}{2}(e^x + 9e^{-x})$  (2)

Hence

(b) solve  $f(x) = 5$  (4)

(c) show that  $\int_{\frac{1}{2}\ln 3}^{\ln 3} \frac{1}{5 \cosh x - 4 \sinh x} dx = \frac{\pi}{18}$  (5)

a)  $f(x) = 5 \cosh x - 4 \sinh x$

$$= \frac{5}{2}(e^x + e^{-x}) - \frac{4}{2}(e^x - e^{-x})$$

$$= \frac{1}{2}e^x + \frac{9}{2}e^{-x}$$

$$= \frac{1}{2}(e^x + 9e^{-x})$$

b)

$$f(x) = 5 \Rightarrow \frac{1}{2}(e^x + 9e^{-x}) = 5$$

$$e^x + 9e^{-x} = 10$$

$$e^{2x} + 9 = 10e^x$$

$$e^{2x} - 10e^x + 9 = 0$$

$$(e^x - 1)(e^x - 9) = 0$$

$$e^x = 1 \quad \text{or} \quad e^x = 9$$

$$x = 0 \quad \text{or} \quad x = \ln 9$$



Question 7 continued

c)

$$\int_{\frac{1}{2}\ln 3}^{\ln 3} \frac{1}{5\cosh x - 4\sinh x} dx$$

$$= \int_{\frac{1}{2}\ln 3}^{\ln 3} \frac{2}{e^x + 9e^{-x}} dx$$

$$= \int_{\frac{1}{2}\ln 3}^{\ln 3}$$

$$L.C.L. u = e^{2x}$$

$$\frac{du}{dx} = e^{2x}$$

$$du = e^{2x} dx$$

$$\int \frac{2e^x}{e^{2x} + 9} dx$$

$$\int_3^{\sqrt{3}} \frac{2 du}{u^2 + 3^2}$$

$$= \left[ \frac{2}{3} \tan^{-1} \left( \frac{u}{3} \right) \right]_3^{\sqrt{3}}$$

$$= \frac{2}{3} \left[ \tan^{-1} 1 - \tan^{-1} \frac{1}{\sqrt{3}} \right]$$

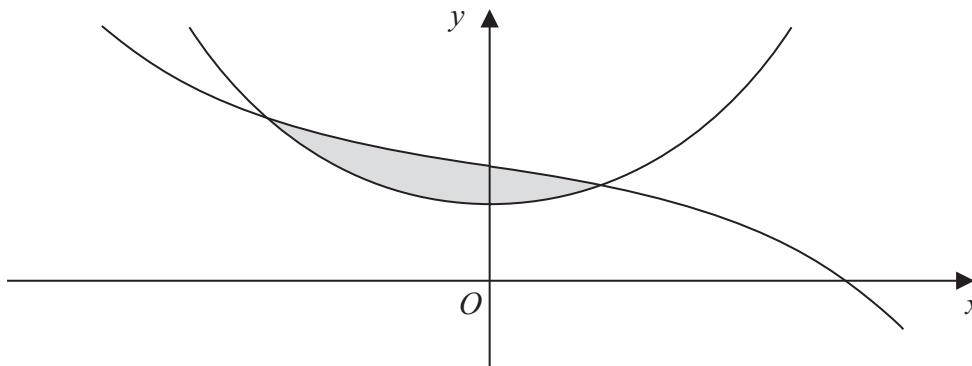
$$= \frac{2}{3} \left[ \frac{\pi}{4} - \frac{\pi}{3} \right]$$

$$= \frac{2}{3} \left[ \frac{\pi}{12} \right]$$

$$= \frac{\pi}{18}$$



7.

**Figure 1**

The curves shown in Figure 1 have equations

$$y = 6 \cosh x \text{ and } y = 9 - 2 \sinh x$$

- (a) Using the definitions of  $\sinh x$  and  $\cosh x$  in terms of  $e^x$ , find exact values for the  $x$ -coordinates of the two points where the curves intersect.

(6)

The finite region between the two curves is shown shaded in Figure 1.

- (b) Using calculus, find the area of the shaded region, giving your answer in the form  $a \ln b + c$ , where  $a$ ,  $b$  and  $c$  are integers.

(6)

$$y = 6 \times \frac{1}{2}(e^x + e^{-x})$$

$$y = 9 - 2 \times \frac{1}{2}(e^x - e^{-x})$$

Intersect when

$$3e^x + 3e^{-x} = 9 - e^x + e^{-x}$$

$$4e^x + 2e^{-x} - 9 = 0$$

$$4e^{2x} - 9e^x + 2 = 0$$

$$(4e^x - 1)(e^x - 2) = 0$$

$$4e^x = 1 \quad \text{or} \quad e^x = 2$$

$$x = \ln\left(\frac{1}{4}\right) \quad x = \ln 2$$



Question 7 continued

b)  $\text{Area} = \int_{\ln \frac{1}{4}}^{\ln 2} (9 - 2 \sinh x - 6 \cosh x) dx$

$$= \left[ 9x - 2 \cosh x - 6 \sinh x \right]_{\ln \frac{1}{4}}^{\ln 2}$$

$$= \left( 9 \ln 2 - \frac{5}{2} - \frac{9}{2} \right) - \left( 9 \ln \left( \frac{1}{4} \right) - \frac{17}{4} + \frac{45}{4} \right)$$

$$= 9 \ln 2 - 7 + 9 \ln 4 - 7$$

$$= 9 \ln 2 + 18 \ln 2 - 14$$

$$= 27 \ln 2 - 14$$


---



## Further Pure Mathematics FP3

Candidates sitting FP3 may also require those formulae listed under Further Pure Mathematics FP1, and Core Mathematics C1–C4.

### Vectors

The resolved part of  $\mathbf{a}$  in the direction of  $\mathbf{b}$  is  $\frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|}$

The point dividing  $AB$  in the ratio  $\lambda : \mu$  is  $\frac{\mu\mathbf{a} + \lambda\mathbf{b}}{\lambda + \mu}$

$$\text{Vector product: } \mathbf{a} \times \mathbf{b} = |\mathbf{a}| |\mathbf{b}| \sin \theta \hat{\mathbf{n}} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{pmatrix}$$

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})$$

If  $A$  is the point with position vector  $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$  and the direction vector  $\mathbf{b}$  is given by  $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ , then the straight line through  $A$  with direction vector  $\mathbf{b}$  has cartesian equation

$$\frac{x - a_1}{b_1} = \frac{y - a_2}{b_2} = \frac{z - a_3}{b_3} (= \lambda)$$

The plane through  $A$  with normal vector  $\mathbf{n} = n_1\mathbf{i} + n_2\mathbf{j} + n_3\mathbf{k}$  has cartesian equation

$$n_1x + n_2y + n_3z + d = 0 \text{ where } d = -\mathbf{a} \cdot \mathbf{n}$$

The plane through non-collinear points  $A$ ,  $B$  and  $C$  has vector equation

$$\mathbf{r} = \mathbf{a} + \lambda(\mathbf{b} - \mathbf{a}) + \mu(\mathbf{c} - \mathbf{a}) = (1 - \lambda - \mu)\mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$$

The plane through the point with position vector  $\mathbf{a}$  and parallel to  $\mathbf{b}$  and  $\mathbf{c}$  has equation

$$\mathbf{r} = \mathbf{a} + s\mathbf{b} + t\mathbf{c}$$

The perpendicular distance of  $(\alpha, \beta, \gamma)$  from  $n_1x + n_2y + n_3z + d = 0$  is  $\frac{|n_1\alpha + n_2\beta + n_3\gamma + d|}{\sqrt{n_1^2 + n_2^2 + n_3^2}}$ .

### **Hyperbolic functions**

$$\cosh^2 x - \sinh^2 x = 1$$

$$\sinh 2x = 2 \sinh x \cosh x$$

$$\cosh 2x = \cosh^2 x + \sinh^2 x$$

$$\operatorname{arccosh} x = \ln \left\{ x + \sqrt{x^2 - 1} \right\} \quad (x \geq 1)$$

$$\operatorname{arsinh} x = \ln \left\{ x + \sqrt{x^2 + 1} \right\}$$

$$\operatorname{artanh} x = \frac{1}{2} \ln \left( \frac{1+x}{1-x} \right) \quad (|x| < 1)$$

### **Conics**

	<b>Ellipse</b>	<b>Parabola</b>	<b>Hyperbola</b>	<b>Rectangular Hyperbola</b>
Standard Form	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	$y^2 = 4ax$	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	$xy = c^2$
Parametric Form	$(a \cos \theta, b \sin \theta)$	$(at^2, 2at)$	$(a \sec \theta, b \tan \theta)$ $(\pm a \cosh \theta, b \sinh \theta)$	$\left( ct, \frac{c}{t} \right)$
Eccentricity	$e < 1$ $b^2 = a^2(1 - e^2)$	$e = 1$	$e > 1$ $b^2 = a^2(e^2 - 1)$	$e = \sqrt{2}$
Foci	$(\pm ae, 0)$	$(a, 0)$	$(\pm ae, 0)$	$(\pm \sqrt{2}c, \pm \sqrt{2}c)$
Directrices	$x = \pm \frac{a}{e}$	$x = -a$	$x = \pm \frac{a}{e}$	$x + y = \pm \sqrt{2}c$
Asymptotes	none	none	$\frac{x}{a} = \pm \frac{y}{b}$	$x = 0, y = 0$

## Differentiation

$f(x)$	$f'(x)$
$\arcsin x$	$\frac{1}{\sqrt{1-x^2}}$
$\arccos x$	$-\frac{1}{\sqrt{1-x^2}}$
$\arctan x$	$\frac{1}{1+x^2}$
$\sinh x$	$\cosh x$
$\cosh x$	$\sinh x$
$\tanh x$	$\operatorname{sech}^2 x$
$\operatorname{arsinh} x$	$\frac{1}{\sqrt{1+x^2}}$
$\operatorname{arcosh} x$	$\frac{1}{\sqrt{x^2-1}}$
$\operatorname{artanh} x$	$\frac{1}{1-x^2}$

## Integration (+ constant; $a > 0$ where relevant)

$f(x)$	$\int f(x) \, dx$
$\sinh x$	$\cosh x$
$\cosh x$	$\sinh x$
$\tanh x$	$\ln \cosh x$
$\frac{1}{\sqrt{a^2 - x^2}}$	$\arcsin \left( \frac{x}{a} \right) \quad ( x  < a)$
$\frac{1}{a^2 + x^2}$	$\frac{1}{a} \arctan \left( \frac{x}{a} \right)$
$\frac{1}{\sqrt{x^2 - a^2}}$	$\operatorname{arcosh} \left( \frac{x}{a} \right), \quad \ln \left\{ x + \sqrt{x^2 - a^2} \right\} \quad (x > a)$
$\frac{1}{\sqrt{a^2 + x^2}}$	$\operatorname{arsinh} \left( \frac{x}{a} \right), \quad \ln \left\{ x + \sqrt{x^2 + a^2} \right\}$
$\frac{1}{a^2 - x^2}$	$\frac{1}{2a} \ln \left  \frac{a+x}{a-x} \right  = \frac{1}{a} \operatorname{artanh} \left( \frac{x}{a} \right) \quad ( x  < a)$
$\frac{1}{x^2 - a^2}$	$\frac{1}{2a} \ln \left  \frac{x-a}{x+a} \right $

### *Arc length*

$$s = \int \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \quad (\text{cartesian coordinates})$$

$$s = \int \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \quad (\text{parametric form})$$

### *Surface area of revolution*

$$\begin{aligned} S_x &= 2\pi \int y \ ds = 2\pi \int y \sqrt{\left(1 + \left(\frac{dy}{dx}\right)^2\right)} dx \\ &= 2\pi \int y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \end{aligned}$$

## Further Pure Mathematics FP1

Candidates sitting FP1 may also require those formulae listed under Core Mathematics C1 and C2.

### *Summations*

$$\sum_{r=1}^n r^2 = \frac{1}{6} n(n+1)(2n+1)$$

$$\sum_{r=1}^n r^3 = \frac{1}{4} n^2 (n+1)^2$$

### *Numerical solution of equations*

The Newton-Raphson iteration for solving  $f(x) = 0$ :  $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

### *Conics*

	Parabola	Rectangular Hyperbola
Standard Form	$y^2 = 4ax$	$xy = c^2$
Parametric Form	$(at^2, 2at)$	$\left(ct, \frac{c}{t}\right)$
Foci	$(a, 0)$	Not required
Directrices	$x = -a$	Not required

### *Matrix transformations*

Anticlockwise rotation through  $\theta$  about  $O$ :  $\begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$

Reflection in the line  $y = (\tan \theta)x$ :  $\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$

In FP1,  $\theta$  will be a multiple of  $45^\circ$ .

## Core Mathematics C4

Candidates sitting C4 may also require those formulae listed under Core Mathematics C1, C2 and C3.

### Integration (+ constant)

$f(x)$	$\int f(x) \, dx$
$\sec^2 kx$	$\frac{1}{k} \tan kx$
$\tan x$	$\ln \sec x $
$\cot x$	$\ln \sin x $
$\operatorname{cosec} x$	$-\ln \operatorname{cosec} x + \cot x , \quad \ln \tan(\frac{1}{2}x) $
$\sec x$	$\ln \sec x + \tan x , \quad \ln \tan(\frac{1}{2}x + \frac{1}{4}\pi) $

$$\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx$$

## Core Mathematics C3

Candidates sitting C3 may also require those formulae listed under Core Mathematics C1 and C2.

### Logarithms and exponentials

$$e^{x \ln a} = a^x$$

### Trigonometric identities

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B} \quad (A \pm B \neq (k + \frac{1}{2})\pi)$$

$$\sin A + \sin B = 2 \sin \frac{A+B}{2} \cos \frac{A-B}{2}$$

$$\sin A - \sin B = 2 \cos \frac{A+B}{2} \sin \frac{A-B}{2}$$

$$\cos A + \cos B = 2 \cos \frac{A+B}{2} \cos \frac{A-B}{2}$$

$$\cos A - \cos B = -2 \sin \frac{A+B}{2} \sin \frac{A-B}{2}$$

### Differentiation

$f(x)$	$f'(x)$
$\tan kx$	$k \sec^2 kx$
$\sec x$	$\sec x \tan x$
$\cot x$	$-\operatorname{cosec}^2 x$
$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$
$\frac{f(x)}{g(x)}$	$\frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$

## Core Mathematics C2

Candidates sitting C2 may also require those formulae listed under Core Mathematics C1.

Cosine rule

$$a^2 = b^2 + c^2 - 2bc \cos A$$

Binomial series

$$(a+b)^n = a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \dots + \binom{n}{r} a^{n-r} b^r + \dots + b^n \quad (n \in \mathbb{N})$$

$$\text{where } \binom{n}{r} = {}^n C_r = \frac{n!}{r!(n-r)!}$$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1 \times 2} x^2 + \dots + \frac{n(n-1)\dots(n-r+1)}{1 \times 2 \times \dots \times r} x^r + \dots \quad (|x| < 1, n \in \mathbb{R})$$

Logarithms and exponentials

$$\log_a x = \frac{\log_b x}{\log_b a}$$

Geometric series

$$u_n = ar^{n-1}$$

$$S_n = \frac{a(1-r^n)}{1-r}$$

$$S_\infty = \frac{a}{1-r} \text{ for } |r| < 1$$

Numerical integration

The trapezium rule:  $\int_a^b y \, dx \approx \frac{1}{2} h \{(y_0 + y_n) + 2(y_1 + y_2 + \dots + y_{n-1})\}$ , where  $h = \frac{b-a}{n}$

# Core Mathematics C1

## Mensuration

$$\text{Surface area of sphere} = 4\pi r^2$$

$$\text{Area of curved surface of cone} = \pi r \times \text{slant height}$$

## Arithmetic series

$$u_n = a + (n - 1)d$$

$$S_n = \frac{1}{2} n(a + l) = \frac{1}{2} n[2a + (n - 1)d]$$