

Integrating Factors for 1st order differential equations

Solve $\frac{dy}{dx} + P(x)y = Q(x)$

by multiplying eqn by the integrating factor $e^{\int P(x)dx}$

Exercise 7A

3a) $x \frac{dy}{dx} + y = \cos x$

$$\frac{dy}{dx} + \frac{y}{x} = \frac{\cos x}{x}$$

$$x \frac{dy}{dx} + y = \cos x$$

$$\frac{d}{dx}(xy) = \cos x$$

$$xy = \sin x + C$$

$$y = \frac{\sin x + C}{x}$$

3b) $e^{-x} \frac{dy}{dx} - e^{-x}y = xe^{-x}$

$$\frac{dy}{dx} - y = xe^{2x}$$

$$e^{-x} \frac{dy}{dx} - e^{-x}y = xe^{-x}$$

$$\frac{d}{dx}(e^{-x}y) = xe^{-x}$$

Integrating factor

$$= e^{\int \frac{1}{x} dx}$$

$$= e^{\ln x}$$

$$= x$$

IF

$$= e^{\int -1 dx}$$

$$= e^{-x}$$

$$e^{-x} y = \int x e^x dx$$

$$\left[\begin{array}{l} \int x e^x dx \quad \text{Let } u = x \quad \text{Let } \frac{du}{dx} = e^x \\ \quad \quad \quad \frac{du}{dx} = 1 \quad \quad \quad u = e^x \\ \int x e^x dx = x e^x - \int e^x dx \\ \quad \quad \quad = x e^x - e^x + C \end{array} \right]$$

$$e^{-x} y = x e^x - e^x + C$$

$$y = (x-1) e^{2x} + C e^x$$

$$3c) \quad \sin x \frac{dy}{dx} + y \cos x = 3$$

$$\frac{d}{dx} (y \sin x) = 3$$

$$y \sin x = 3x + C$$

$$y = \frac{3x + C}{\sin x}$$

$$3d) \quad \frac{1}{x} \frac{dy}{dx} - \frac{1}{x^2} y = e^x$$

$$\frac{d}{dx} \left(\frac{1}{x} y \right) = e^x$$

$$\frac{1}{x} y = e^x + c$$
$$y = x e^x + c x$$

$$3e) \quad x^2 e^y \frac{dy}{dx} + 2x e^y = x$$

$$\frac{d}{dx} (x^2 e^y) = x$$

$$x^2 e^y = \frac{x^2}{2} + c$$

$$e^y = \frac{1}{2} + \frac{c}{x^2}$$

$$y = \ln\left(\frac{1}{2} + \frac{c}{x^2}\right)$$

$$3f) \quad 4xy \frac{dy}{dx} + 2y^2 = x^2$$

$$\frac{d}{dx} (2xy^2) = x^2$$

$$2xy^2 = \frac{x^3}{3} + c$$

$$y^2 = \frac{x^2}{6} + \frac{c}{2x}$$

$$y = \pm \sqrt{\frac{x^2}{6} + \frac{c}{2x}}$$

7a) $\frac{dy}{dx} + 2y = e^{2x}$

$\begin{aligned} & \text{I.F.} \\ &= e^{\int 2 dx} \\ &= e^{2x} \end{aligned}$

$e^{2x} \frac{dy}{dx} + 2e^{2x}y = e^{3x}$

$\frac{d}{dx}(e^{2x}y) = e^{3x}$

$e^{2x}y = \frac{1}{3}e^{3x} + C$

$y = \frac{1}{3}e^{x} + Ce^{-2x}$

7b) $\frac{dy}{dx} + y \cot x = 1$

$\begin{aligned} & \text{I.F.} \\ &= e^{\int \frac{\cot x}{\sin x} dx} \\ &= e^{\ln |\sin x|} \\ &= \sin x \end{aligned}$

$\sin x \frac{dy}{dx} + y \cos x = \sin x$

$\frac{d}{dx}(y \sin x) = \sin x$

$y \sin x = -\cos x + C$

$y = -\cot x + C \csc x$

Homework - the rest of Q7