Infinite series C and S are defined by

$$C = \cos 2\theta - \tfrac{1}{2}\cos 5\theta + \tfrac{1}{4}\cos 8\theta - \tfrac{1}{8}\cos 11\theta + \ldots,$$

$$S = \sin 2\theta - \tfrac{1}{2}\sin 5\theta + \tfrac{1}{4}\sin 8\theta - \tfrac{1}{8}\sin 11\theta + \ldots.$$

(iii) Show that $C = \dfrac{4\cos 2\theta + 2\cos\theta}{5 + 4\cos 3\theta}$, and find a similar expression for S. [8]

$\cos\theta + i\sin\theta = e^{i\theta}$

$C + iS = e^{i2\theta} - \frac{1}{2}e^{i5\theta} + \frac{1}{4}e^{i8\theta} - \frac{1}{8}e^{i11\theta} + \ldots$

This is a G.P.

$a + ar^2 + ar^3 + \ldots$ $\quad S_n = \dfrac{a(1-r^n)}{1-r}$

$S_\infty = \dfrac{a}{1-r}$

$a = e^{i2\theta} \qquad r = -\frac{1}{2}e^{i3\theta}$

$C + iS = S_\infty = \dfrac{e^{2i\theta}}{1 + \frac{1}{2}e^{i3\theta}}$

If $w = \frac{1}{2}e^{3i\theta}$ show $(1+w)(1+w^*) = \frac{5}{4} + \cos 3\theta$

$\left(1 + \frac{1}{2}e^{3i\theta}\right)\left(1 + \frac{1}{2}e^{-3i\theta}\right)$

$= 1 + \frac{1}{2}e^{3i\theta} + \frac{1}{2}e^{-3i\theta} + \frac{1}{4}e^{0}$

$= \frac{5}{4} + \frac{1}{2}\left(e^{3i\theta} + e^{-3i\theta}\right)$

$= \frac{5}{4} + \cos 3\theta$

$$C+iS = \frac{e^{2i\theta}}{\left(1+\frac{1}{2}e^{i3\theta}\right)} \times \frac{\left(1+\frac{1}{2}e^{-3i\theta}\right)}{\left(1+\frac{1}{2}e^{-3i\theta}\right)}$$

$$= \frac{e^{2i\theta} + \frac{1}{2}e^{-i\theta}}{\frac{5}{4} + \cos 3\theta}$$

$$= \frac{\cos 2\theta + i\sin 2\theta + \frac{1}{2}\cos\theta - \frac{1}{2}\sin\theta}{\frac{5}{4} + \cos 3\theta}$$

$$C+iS = \frac{4\cos 2\theta + 2\cos\theta + i(4\sin 2\theta - 2\sin\theta)}{5 + 4\cos 3\theta}$$

Equate Re and Im Parts

$$C = \frac{4\cos 2\theta + 2\cos\theta}{5 + 4\cos 3\theta}$$

$$S = \frac{4\sin 2\theta - 2\sin\theta}{5 + 4\cos 3\theta}$$

(b) In this part of the question, n is a positive integer and θ is a real number with $0 < \theta < \frac{\pi}{n}$.

(i) Express $e^{-\frac{1}{2}j\theta} + e^{\frac{1}{2}j\theta}$ in simplified trigonometric form, and hence, or otherwise, show that

$$1 + e^{j\theta} = 2e^{\frac{1}{2}j\theta}\cos\tfrac{1}{2}\theta. \qquad [4]$$

Series C and S are defined by

$$C = 1 + \binom{n}{1}\cos\theta + \binom{n}{2}\cos 2\theta + \binom{n}{3}\cos 3\theta + \ldots + \binom{n}{n}\cos n\theta,$$
$$S = \binom{n}{1}\sin\theta + \binom{n}{2}\sin 2\theta + \binom{n}{3}\sin 3\theta + \ldots + \binom{n}{n}\sin n\theta.$$

(ii) Find C and S, and show that $\frac{S}{C} = \tan\tfrac{1}{2}n\theta$. [7]

$$e^{-\frac{1}{2}i\alpha} + e^{\frac{1}{2}i\alpha} = 2\cos\frac{\alpha}{2}$$

$$(1 + e^{i\alpha}) = e^{\frac{i\alpha}{2}}\left(e^{-\frac{i\alpha}{2}} + e^{\frac{i\alpha}{2}}\right)$$

$$= 2e^{\frac{i\alpha}{2}}\cos\frac{\alpha}{2}$$

$$C + iS = 1 + \binom{n}{1}(\cos\alpha + i\sin\alpha) + \binom{n}{2}(\cos 2\alpha + i\sin 2\alpha) + \ldots + \binom{n}{n}(\cos n\alpha + i\sin n\alpha)$$

$$= 1 + \binom{n}{1}e^{i\alpha} + \binom{n}{2}e^{i2\alpha} + \ldots$$

$$+\binom{n}{n} e^{in\theta}$$

$$= (1 + e^{i\theta})^n$$

$$= \left(2 e^{i\frac{\theta}{2}} \cos\frac{\theta}{2}\right)^n$$

$$C + iS = 2^n e^{i\frac{n\theta}{2}} \cos^n\frac{\theta}{2}$$

$$C = 2^n \cos\left(\frac{n\theta}{2}\right) \cos^n\frac{\theta}{2}$$

$$S = 2^n \sin\left(\frac{n\theta}{2}\right) \cos^n\frac{\theta}{2}$$

$$\frac{S}{C} = \frac{\cancel{2^n} \sin\left(\frac{n\theta}{2}\right) \cancel{\cos^n\left(\frac{\theta}{2}\right)}}{\cancel{2^n} \cos\left(\frac{n\theta}{2}\right) \cancel{\cos^n\left(\frac{\theta}{2}\right)}}$$

$$= \tan\left(\frac{n\theta}{2}\right)$$

(b) **(i)** Show that $(1 - 2e^{j\theta})(1 - 2e^{-j\theta}) = 5 - 4\cos\theta$. [3]

Series C and S are defined by

$$C = 2\cos\theta + 4\cos 2\theta + 8\cos 3\theta + \ldots + 2^n \cos n\theta,$$
$$S = 2\sin\theta + 4\sin 2\theta + 8\sin 3\theta + \ldots + 2^n \sin n\theta.$$

(ii) Show that $C = \dfrac{2\cos\theta - 4 - 2^{n+1}\cos(n+1)\theta + 2^{n+2}\cos n\theta}{5 - 4\cos\theta}$, and find a similar expression for S. [9]

i) $(1 - 2e^{i\theta})(1 - 2e^{-i\theta})$

$= 1 - 2e^{i\theta} - 2e^{-i\theta} + 4$

$= 5 - 2(e^{i\theta} + e^{-i\theta})$

$= 5 - 2(2\cos\theta) = 5 - 4\cos\theta$

$C + iS = 2(\cos\theta + i\sin\theta) + 4(\cos 2\theta + i\sin 2\theta)$
$+ 8(\cos 3\theta + i\sin 3\theta) + \ldots + 2^n(\cos n\theta + i\sin n\theta)$

$= 2e^{i\theta} + 4e^{2i\theta} + 8e^{3i\theta} + \ldots + 2^n e^{ni\theta}$

$=$ GP n terms

$a = 2e^{i\theta}$

$r = 2e^{i\theta}$

$C + iS = \dfrac{a(1 - r^n)}{1 - r} = \dfrac{2e^{i\theta}(1 - 2^n e^{in\theta})}{1 - 2e^{i\theta}}$

$$C+iS = \frac{2e^{i\theta}\left(1-2^n e^{in\theta}\right)}{1-2e^{i\theta}} \times \frac{1-2e^{-i\theta}}{1-2e^{-i\theta}}$$

$$= \frac{\left(2e^{i\theta}-2^{n+1}e^{i(n+1)\theta}\right)\left(1-2e^{-i\theta}\right)}{5-4\cos\theta}$$

$$= \frac{2e^{i\theta}-2^{n+1}e^{i(n+1)\theta}-4-2^{n+2}e^{in\theta}}{5-4\cos\theta}$$

Equating Re and Im parts

$$C = \frac{2\cos\theta-2^{n+1}\cos(n+1)\theta-4+2^{n+2}\cos n\theta}{5-4\cos\theta}$$

$$S = \frac{2\sin\theta-2^{n+1}\sin(n+1)\theta-2^{n+2}\sin n\theta}{5-4\cos\theta}$$