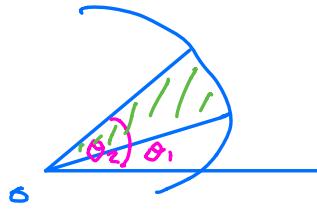


Area Enclosed by a Polar Curve

$$\text{Area} = \frac{1}{2} \int_{\theta_1}^{\theta_2} r^2 d\theta$$



$$\text{Ex1} \quad r = a \cos 3\theta$$

Find area in loop from $\theta = -\frac{\pi}{6}$ to $\theta = \frac{\pi}{6}$

$$\begin{aligned}\text{Area} &= \frac{1}{2} \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} a^2 \cos^2 3\theta d\theta \\ &= \frac{a^2}{2} \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \frac{1 + \cos 6\theta}{2} d\theta \\ &= \frac{a^2}{4} \left[\theta + \frac{1}{6} \sin 6\theta \right]_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \\ &= \frac{a^2}{4} \left[\left(\frac{\pi}{6} + \frac{1}{6} \sin \pi \right) - \left(-\frac{\pi}{6} + \frac{1}{6} \sin(-\pi) \right) \right] \\ &= \frac{a^2}{4} \left[\frac{\pi}{6} + \frac{\pi}{6} \right] \\ &= \frac{a^2 \pi}{12}\end{aligned}$$

$$\text{Ex2} \quad r = a(\sqrt{2} + 2 \cos \theta)$$

$$\theta_1 = -\frac{3\pi}{4} \quad \theta_2 = \frac{3\pi}{4}$$

$$\begin{aligned}
 \text{Area} &= \frac{1}{2} \int_{-\frac{3\pi}{4}}^{\frac{3\pi}{4}} r^2 d\theta \\
 &= \frac{1}{2} \int_{-\frac{3\pi}{4}}^{\frac{3\pi}{4}} a^2 (\sqrt{2} + 2\cos\theta)^2 d\theta \\
 &= \frac{a^2}{2} \int_{-\frac{3\pi}{4}}^{\frac{3\pi}{4}} (2 + 4\sqrt{2}\cos\theta + 4\cos^2\theta) d\theta \\
 &= \frac{a^2}{2} \int_{-\frac{3\pi}{4}}^{\frac{3\pi}{4}} (2 + 4\sqrt{2}\cos\theta + 2 + 2\cos 2\theta) d\theta \\
 &= \frac{a^2}{2} \left[4\theta + 4\sqrt{2}\sin\theta + \sin 2\theta \right]_{-\frac{3\pi}{4}}^{\frac{3\pi}{4}} \\
 &= \frac{a^2}{2} \left[(3\pi + 4 - 1) - (-3\pi - 4 + 1) \right] \\
 &= \frac{a^2}{2} [6\pi + 6] = 3a^2(\pi + 1)
 \end{aligned}$$

$$\text{Ex 3} \quad r = a e^{-k\theta} \quad 0 \leq \theta \leq \pi$$

$$A = \frac{1}{2} \int_0^\pi r^2 d\theta$$

$$\begin{aligned}
A &= \frac{1}{2} \int_0^{\pi} a^2 e^{-2k\theta} d\theta \\
&= \frac{a^2}{2} \left[\frac{e^{-2k\theta}}{-2k} \right]_0^{\pi} \\
&= -\frac{a^2}{4k} \left[e^{-2k\theta} \right]_0^{\pi} \\
&= -\frac{a^2}{4k} \left[e^{-2k\pi} - e^0 \right] \\
&= \frac{a^2}{4k} (1 - e^{-2k\pi})
\end{aligned}$$

Ex 4 $r = a(1 - \cos\theta) \quad 0 \leq \theta \leq \frac{\pi}{2}$

$$\begin{aligned}
A &= \frac{1}{2} \int_0^{\frac{\pi}{2}} r^2 d\theta \\
&= \frac{1}{2} \int_0^{\frac{\pi}{2}} a^2 (1 - \cos\theta)^2 d\theta \\
&= \frac{a^2}{2} \int_0^{\frac{\pi}{2}} (1 - 2\cos\theta + \cos^2\theta) d\theta \\
&= \frac{a^2}{2} \int_0^{\frac{\pi}{2}} \left(1 - 2\cos\theta + \frac{1 + \cos 2\theta}{2} \right) d\theta \\
&= \frac{a^2}{2} \left[\frac{3\theta}{2} - 2\sin\theta + \frac{\sin 2\theta}{4} \right]_0^{\frac{\pi}{2}}
\end{aligned}$$

$$\begin{aligned}
 &= \frac{a^2}{2} \left[\left(\left(\frac{3\pi}{4} - 2 + 0 \right) - (0 - 0 + 0) \right) \right] \\
 &= \frac{a^2}{2} \left(\frac{3\pi}{4} - 2 \right) \\
 &= \frac{a^2}{8} (3\pi - 8)
 \end{aligned}$$

Exercise 5C

$$1e) \quad r^2 = a^2 \tan \theta \quad 0 \leq \theta \leq \frac{\pi}{4}$$

$$\begin{aligned}
 \text{Area} &= \frac{1}{2} \int_0^{\frac{\pi}{4}} r^2 d\theta \\
 &= \frac{1}{2} \int_0^{\frac{\pi}{4}} a^2 \tan \theta d\theta \\
 &= \frac{a^2}{2} \int_0^{\frac{\pi}{4}} \frac{\sin \theta}{\cos \theta} d\theta \\
 &= \frac{a^2}{2} \left[-\ln |\cos \theta| \right]_0^{\frac{\pi}{4}} \\
 &= \frac{a^2}{2} \left[-\ln \cos \frac{\pi}{4} - -\ln \cos 0 \right] \\
 &= \frac{a^2}{2} \left[-\ln \frac{1}{\sqrt{2}} + \ln 1 \right] \\
 &= \frac{a^2}{2} \left[\ln \sqrt{2} + 0 \right] \\
 &= \frac{a^2}{2} \left[\frac{1}{2} \ln 2 \right] = \frac{a^2 \ln 2}{4}
 \end{aligned}$$
