FM CORE PURE UNIT 6 PROOF – INDUCTION

1) a) Prove $\sum_{r=1}^{n} r(r+1) = \frac{1}{3}n(n+1)(n+2)$

When $n=1$ $\sum_{r=1}^{1} r(r+1) = 1(1+1) = 2$

$\frac{1}{3}n(n+1)(n+2) = \frac{1}{3}\times 1\times 2\times 3 = 2$ ✓

$\therefore$ formula true for $n=1$

Assume true for $n=k$

then $\sum_{r=1}^{k} r(r+1) = \frac{1}{3}k(k+1)(k+2)$

$$\Rightarrow \sum_{r=1}^{k+1} r(r+1) = \frac{1}{3}k(k+1)(k+2) + (k+1)(k+2)$$

$$= \frac{1}{3}(k+1)(k+2)\left[k+3\right]$$

$$= \frac{1}{3}(k+1)\left((k+1)+1\right)\left((k+1)+2\right)$$

This is same formula with k replaced by $k+1$

$\therefore$ if true for $n=k$, it is also true for $n=k+1$

Since true for $n=1$, by mathematical induction it is true for all positive integers n.

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b) $$\sum_{r=1}^{2n-1} r(r+1) = \frac{1}{3}(2n-1)(2n)(2n+1)$$

$$= \frac{1}{3}(2n)(4n^2-1)$$

$$= \frac{2}{3}n(4n^2-1)$$

PROOF - INDUCTION

2) Prove $\sum_{r=1}^{n} \frac{r-1}{r!} = \frac{n!-1}{n!}$

$n=1$ $\quad \sum_{r=1}^{1} \frac{r-1}{r!} = \frac{0}{1} = 0, \quad \frac{n!-1}{n!} = \frac{1-1}{1} = 0$ ✓

$\therefore$ true for $n=1$

Assume true for $n=k$ then $\sum_{r=1}^{k} \frac{r-1}{r!} = \frac{k!-1}{k!}$

$$\Rightarrow \sum_{r=1}^{k+1} \frac{r-1}{r!} = \frac{k!-1}{k!} + \frac{k+1-1}{(k+1)!}$$

$$= \frac{k!-1}{k!} + \frac{k}{(k+1)!}$$

$$= \frac{(k+1)(k!-1)+k}{(k+1)!}$$

$$= \frac{(k+1)! - (k+1) + k}{(k+1)!}$$

$$= \frac{(k+1)! - 1}{(k+1)!}$$

This is same formula with k replaced by $k+1$

$\therefore$ if formula true for $n=k$ it is also true for $n=k+1$

Since it is true for $n=1$, by mathematical induction it is true for all positive integers n.

3) Prove $5^{2n} + 11$ is divisible by 6

$n = 1$ $\quad 5^2 + 11 = 36 = 6 \times 6$ $\quad$ so true for $n=1$

Assume true for $n = k$ then

$$5^{2k} + 11 = 6x \quad \text{for some integer } x$$

$$\Rightarrow \quad 5^{2(k+1)} + 11 = 5^{2k} \times 5^2 + 11$$

$$25 \times 5^{2k} + 11$$

$$= 25 \times 5^{2k} + 25 \times 11 - 24 \times 11$$

$$= 25\left(5^{2k} + 11\right) - 24 \times 11$$

$$= 25(6x) - 6(4 \times 11)$$

$$= 6(25x - 44)$$

which is divisible by 6 since 6 is a factor

$\therefore$ if true for $n = k$, formula is also true for $n = k+1$

Since true for $n = 1$, by mathematical induction it is true for all positive integers n

4) Prove $11^n - 7^n$ is divisible by 4

$n=1$ $\quad 11^1 - 7^1 = 11 - 7 = 4 = 4 \times 1$ ✓

$\therefore$ true for $n=1$

Assume true for $n=k$, then $11^k - 7^k = 4x$

for some integer x

$$\Rightarrow \quad 11^{k+1} - 7^{k+1}$$
$$= 11(11^k) - 7(7^k)$$
$$= 4(11^k) + 7(11^k) - 7(7^k)$$
$$= 4(11^k) + 7(11^k - 7^k)$$
$$= 4(11^k) + 7(4x)$$
$$= 4(11^k + 7x)$$

which is divisible by 4 since 4 is a factor

$\therefore$ if formula true for $n=k$, it is also true for $n=k+1$

Since true for $n=1$, by mathematical induction it is true for all positive integers n.

5) Prove $n^3 + 9n^2 + 5n$ is divisible by 3

$n=1$ $\quad n^3 + 9n^2 + 5n = 1+9+5 = 15 = 5 \times 3$ ✓

so true for $n=1$

Assume true for $n=k$, then $k^3 + 9k^2 + 5k = 3x$

for some integer x

$$\Rightarrow \quad (k+1)^3 + 9(k+1)^2 + 5(k+1)$$

$$= \cancel{k^3} + 3k^2 + 3k + 1 + \cancel{9k^2} + 18k + 9 \cancel{+5k} + 5$$

$$= (k^3 + 9k^2 + 5k) + 3k^2 + 21k + 15$$

$$= 3x + 3(k^2 + 7k + 5)$$

$$= 3(x + k^2 + 7k + 5)$$

which is divisible by 3 since 3 is a factor

$\therefore$ if formula is true for $n=k$ it is also true for $n = k+1$

Since true for $n=1$, by mathematical induction it is true for all positive integers n.

PROOF - INDUCTION

6) Prove $\begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} n+1 & -n \\ n & -(n-1) \end{pmatrix}$

$n=1$ $\quad \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}^1 = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}$ ✓

$\begin{pmatrix} n+1 & -n \\ n & -(n-1) \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}$

formula true when $n=1$

Assume true for $n=k$ then $\begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}^k = \begin{pmatrix} k+1 & -k \\ k & -(k-1) \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}^{k+1} = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}^k \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} k+1 & -k \\ k & -(k-1) \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 2(k+1)-k & -(k+1) \\ 2k-(k-1) & -k \end{pmatrix}$$

$$= \begin{pmatrix} (k+1)+1 & -(k+1) \\ k+1 & -((k+1)-1) \end{pmatrix}$$

This is same formula with k replaced by $k+1$

$\therefore$ if true for $n=k$ it is also true for $n=k+1$

Since true for $n=1$, by mathematical induction it is true for all positive integers n

7) a) Prove $\begin{pmatrix} 3 & 0 \\ 1 & 1 \end{pmatrix}^n = \begin{pmatrix} 3^n & 0 \\ \frac{1}{2}(3^n-1) & 1 \end{pmatrix}$

$n=1$ $\begin{pmatrix} 3 & 0 \\ 1 & 1 \end{pmatrix}^1 = \begin{pmatrix} 3 & 0 \\ 1 & 1 \end{pmatrix}$ $\begin{pmatrix} 3^n & 0 \\ \frac{1}{2}(3^n-1) & 1 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 1 & 1 \end{pmatrix}$ ✓

formula true for $n=1$

Assume true for $n=k$ then

$$\begin{pmatrix} 3 & 0 \\ 1 & 1 \end{pmatrix}^k = \begin{pmatrix} 3^k & 0 \\ \frac{1}{2}(3^k-1) & 1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 3 & 0 \\ 1 & 1 \end{pmatrix}^{k+1} = \begin{pmatrix} 3 & 0 \\ 1 & 1 \end{pmatrix}^k \begin{pmatrix} 3 & 0 \\ 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 3^k & 0 \\ \frac{1}{2}(3^k-1) & 1 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 3^{k+1} & 0 \\ \frac{3}{2}(3^k-1)+1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 3^{k+1} & 0 \\ \frac{1}{2}(3^{k+1}-3)+1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 3^{k+1} & 0 \\ \frac{1}{2}(3^{k+1}-1) & 1 \end{pmatrix}$$

This is same formula with k replaced by $k+1$

$\therefore$ if true for $n=k$ it is also true for $n=k+1$

Since true for $n=1$, by mathematical induction it is true for all positive integers n.

7b) $\underline{M} = \begin{pmatrix} 3 & 0 \\ 1 & 1 \end{pmatrix}$ $\quad \underline{M}^n = \begin{pmatrix} 3^n & 0 \\ \frac{1}{2}(3^n-1) & 1 \end{pmatrix}$

$$\left(\underline{M}^n\right)^{-1} = \frac{1}{3^n - 0}\begin{pmatrix} 1 & 0 \\ -\frac{1}{2}(3^n-1) & 3^n \end{pmatrix}$$

$$= \frac{1}{3^n}\begin{pmatrix} 1 & 0 \\ \frac{1}{2}(1-3^n) & 3^n \end{pmatrix}$$

OR $\begin{pmatrix} 3^{-n} & 0 \\ \frac{1}{2}(3^{-n}-1) & 1 \end{pmatrix}$