FM CORE PURE UNIT 2 MATRICES

1) $\underline{M} = \begin{pmatrix} 2 & p+4 \\ -p & 3 \end{pmatrix}$

Show non-singular for all p

If singular then $\det \underline{M} = 0$

$\Rightarrow \quad 2 \times 3 - (p+4)(-p) = 0$

$6 + p^2 + 4p = 0$

$p^2 + 4p + 6 = 0$

Discriminant $= 4^2 - 4 \times 1 \times 6$

$= 16 - 24$

$= -8 \quad \therefore$ no real roots

$\therefore$ no value of p for which $\underline{M}$ is singular

2) $\underline{M} = \begin{pmatrix} a & 21 \\ b & -8 \end{pmatrix} \qquad \underline{M}^2 = \underline{I}$

$\begin{pmatrix} a & 21 \\ b & -8 \end{pmatrix}\begin{pmatrix} a & 21 \\ b & -8 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$a^2 + 21b = 1 \quad (1)$

$21a - 168 = 0 \quad (2)$

From (2) $\quad 21a = 168$

$a = \frac{168}{21} = 8$

Sub in (1) $\quad 8^2 + 21b = 1$

$64 + 21b = 1$

$21b = -63$

$b = -\frac{63}{21} = -3$

$a = 8, \quad b = -3$

Check

$\begin{pmatrix} 8 & 21 \\ -3 & -8 \end{pmatrix}\begin{pmatrix} 8 & 21 \\ -3 & -8 \end{pmatrix} = \begin{pmatrix} 64-63 & 168-168 \\ -24+24 & -63+64 \end{pmatrix}$

$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \checkmark$

3) $\underline{M} = \begin{pmatrix} -\frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix}$

a) Rotation by θ anti-clockwise is $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

$\Rightarrow \quad \theta = \frac{5\pi}{6}$ or 150°

Rotation about $(0,0)$ by 150° anti-clockwise

b) $P' = \begin{pmatrix} -\frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$= \left(-\frac{\sqrt{3}}{2} - 1, \ \frac{1}{2} - \sqrt{3}\right)$

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3b cont) $Q' = \begin{pmatrix} -\frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix}\begin{pmatrix} 2 \\ 5 \end{pmatrix}$

$= \left(-\sqrt{3} - \frac{5}{2}, 1 - \frac{5\sqrt{3}}{2}\right)$

c) $\underline{M}^2$ would represent a rotation about $(0,0)$ by $300°$ anti-clockwise or $60°$ clockwise

$$\underline{M}^2 = \begin{pmatrix} \cos(-60) & -\sin(-60) \\ \sin(-60) & \cos(-60) \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$$

4) $\underline{A} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

a) $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ -x \end{pmatrix}$

Reflection in line $y = -x$

b) $(x, y) \rightarrow (-y, -x)$

$a = -(5+b)$ (1)

$b = -(4+2a)$ (2)

From (1)

$a = -5 - b$

$a + b = -5$ (3)

From (2) $b = -4 - 2a$

$\Rightarrow a + (-4 - 2a) = -5$

$a - 4 - 2a = -5$

$-a = -1$

$a = 1$

Sub for a

$1 + b = -5$

$b = -6$

5) $\underline{P} = \begin{pmatrix} q & \sqrt{5} \\ \sqrt{5} & -q \end{pmatrix}$

$$\underline{P}^2 = \begin{pmatrix} q & \sqrt{5} \\ \sqrt{5} & -q \end{pmatrix}\begin{pmatrix} q & \sqrt{5} \\ \sqrt{5} & -q \end{pmatrix}$$

$$= \begin{pmatrix} q^2+5 & q\sqrt{5} - q\sqrt{5} \\ q\sqrt{5} - q\sqrt{5} & q^2+5 \end{pmatrix}$$

$$= \begin{pmatrix} q^2+5 & 0 \\ 0 & q^2+5 \end{pmatrix}$$

$$= (q^2+5)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

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5 cont Enlargement about $(0,0)$

scale factor (q^2+5)

6) a) $\underline{M}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ -y \\ z \end{pmatrix}$

$\underline{M} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

b) Midpoint of $(2,5,9)$ and $(8,5,9)$

$= (5,5,9)$

Image $(5,-5,9)$

c) Simply reflect again in plane $y=0$ to get back to original

so $\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

7) a) $\underline{M} = \begin{pmatrix} -1 & -\sqrt{2} \\ \sqrt{2} & -1 \end{pmatrix}$

$\det \underline{M} = (-1)(-1) - (\sqrt{2})(-\sqrt{2})$

$= 1+2$

$= 3 \neq 0 \therefore$ non-singular

7b) $\det \underline{M} = 3$

so S has area 20×3

$= 60$ sq units

7c) Rotation $\theta°$ anti-clockwise

$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

If enlargement s.f k

then $\begin{pmatrix} k\cos\theta & -k\sin\theta \\ k\sin\theta & k\cos\theta \end{pmatrix}$

$\Rightarrow k\cos\theta = -1$

$k\sin\theta = \sqrt{2}$

$k^2\cos^2\theta = 1$

$k^2\sin^2\theta = 2$

$k^2(\cos^2\theta + \sin^2\theta) = 3$

$k^2 = 3$

$k = \sqrt{3}$

d) $\cos\theta = -\dfrac{1}{\sqrt{3}}$, $\sin\theta = \dfrac{\sqrt{2}}{\sqrt{3}}$

$\theta = \cos^{-1}\left(-\frac{1}{\sqrt{3}}\right)$

$\theta = 125.3°$

8) a) $$\underline{M} = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} -\frac{1}{2}x + \frac{\sqrt{3}}{2}y \\ -\frac{\sqrt{3}}{2}x - \frac{1}{2}y \\ z \end{pmatrix}$$

$\cos\theta = -\frac{1}{2}$
$\sin\theta = -\frac{\sqrt{3}}{2}$

Rotation by 120° clockwise about z-axis

b) $$\begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} -p \\ 0 \\ p \end{pmatrix}$$

$$\left(+\frac{p}{2}, \frac{\sqrt{3}p}{2}, p\right)$$

9) Let h be housing (residential (area)
c commercial
r recreation

$$h + c + r = 200$$

$$r = c + 20$$

$$\Rightarrow \quad -c + r = 20$$

$$1.22h + 0.9c + 1.28r = 240$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & 1 \\ 1.22 & 0.9 & 1.28 \end{pmatrix}\begin{pmatrix} h \\ c \\ r \end{pmatrix} = \begin{pmatrix} 200 \\ 20 \\ 240 \end{pmatrix}$$

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Let $$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & 1 \\ 1.22 & 0.9 & 1.28 \end{pmatrix}$$

$$A\begin{pmatrix} h \\ c \\ r \end{pmatrix} = \begin{pmatrix} 200 \\ 20 \\ 240 \end{pmatrix}$$

$$\begin{pmatrix} h \\ c \\ r \end{pmatrix} = A^{-1}\begin{pmatrix} 200 \\ 20 \\ 240 \end{pmatrix}$$

By calculator

$$h = 140$$
$$c = 20$$
$$r = 40$$

In 2001

residential 140 sq km

commercial 20 sq km

recreational 40 sq km

10) a) $$\underline{M} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

Rotation about (0,0) 135° anti-clockwise

10 b) $$\underline{M}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -\sqrt{2} \\ 3\sqrt{2} \end{pmatrix}$$

$$\begin{pmatrix} a \\ b \end{pmatrix} = \underline{M}^{-1}\begin{pmatrix} -\sqrt{2} \\ 3\sqrt{2} \end{pmatrix}$$

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$$

$$a = 4, \quad b = -2$$