COMPLEX NUMBERS — SERIES — EXERCISE 1E

1) $z = e^{\frac{\pi i}{n}}$

a) $1 + z + z^2 + \dots + z^{2n-1}$

GP $a = 1$, $r = e^{\frac{\pi i}{n}}$ terms $= 2n$

$$\Sigma = \frac{a(1-r^{2n})}{1-r} = \frac{1\left(1-\left(e^{\frac{\pi i}{n}}\right)^{2n}\right)}{1-e^{\frac{\pi i}{n}}}$$

$$= \frac{1(1-e^{2\pi i})}{1-e^{\frac{\pi i}{n}}}$$

$$= \frac{1(1-1)}{1-e^{\frac{\pi i}{n}}} = 0$$

2) $z = e^{\frac{\pi i}{2}}$ $\quad \sum_{r=0}^{12} z^r$

GP $a = 1$, $r = e^{\frac{\pi i}{2}}$, $n = 13$

$$\sum_{r=0}^{12} z^r = \frac{a(1-r^n)}{1-r} = \frac{1\left(1-e^{\frac{13\pi i}{2}}\right)}{1-e^{\frac{\pi}{2}i}}$$

$$= \frac{1\left(1-e^{\frac{\pi}{2}i}\right)}{\left(1-e^{\frac{\pi i}{2}}\right)} = 1$$

3) $$\sum_{r=0}^{7} (1+i)^r = \sum_{r=0}^{7} \left(\sqrt{2}\, e^{i\frac{\pi}{4}}\right)^r$$

GP $a = 1$, $r = \sqrt{2}\, e^{i\frac{\pi}{4}}$, $n = 8$

$$\Sigma = \frac{a(1-r^n)}{1-r} = \frac{1(1-16e^{i2\pi})}{1-\sqrt{2}\, e^{i\frac{\pi}{4}}}$$

$$= \frac{-15}{1-(1+i)} = \frac{-15}{i} = 15i$$

1b) $z = e^{\frac{\pi i}{n}}$ $\qquad 1 + z + z^2 + + z^n$

GP $\quad a = 1, \quad r = e^{\frac{\pi i}{n}}, \quad$ terms $= n+1$

$$\Sigma = \frac{a(1-r^{n+1})}{1-r} = \frac{1\left(1-e^{\frac{\pi i}{n}(n+1)}\right)}{1-e^{\frac{\pi i}{n}}}$$

$$= \frac{1-e^{\pi i}.e^{\frac{\pi i}{n}}}{1-e^{\frac{\pi i}{n}}}$$

$$= \frac{1+e^{\frac{\pi i}{n}}}{1-e^{\frac{\pi i}{n}}}$$

$$= \frac{\left(1+e^{\frac{\pi i}{n}}\right)e^{-\frac{\pi i}{2n}}}{e^{-\frac{\pi i}{2n}}-e^{\frac{\pi i}{2n}}}$$

$$= \frac{e^{-\frac{\pi i}{2n}}+e^{\frac{\pi i}{2n}}}{-\left(e^{\frac{\pi i}{2n}}-e^{-\frac{\pi i}{2n}}\right)}$$

$$= \frac{2\cos\frac{\pi}{2n}}{-2i\sin\frac{\pi}{2n}}$$

$$= i\cot\frac{\pi}{2n}$$

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4) $C = 1 + \frac{1}{3}\cos\theta + \frac{1}{9}\cos 2\theta + \ldots$

$S = \quad \frac{1}{3}\sin\theta + \frac{1}{9}\sin 2\theta + \ldots$

a) $C + iS = 1 + \frac{1}{3}e^{i\theta} + \frac{1}{9}e^{i2\theta}$

Infinite GP $\quad a = 1 \quad r = \frac{1}{3}e^{i\theta}$

$$S_\infty = C + iS = \frac{a}{1-r} = \frac{1}{1-\frac{1}{3}e^{i\theta}}$$

$$= \frac{3}{3-e^{i\theta}}$$

4 cont

$$C + iS = \frac{3}{3 - e^{i\theta}} = \frac{3}{3 - \cos\theta - i\sin\theta}$$

$$= \frac{3}{(3-\cos\theta) - i\sin\theta} \times \frac{(3-\cos\theta) + i\sin\theta}{(3-\cos\theta) + i\sin\theta}$$

$$= \frac{9 - 3\cos\theta + 3i\sin\theta}{(3-\cos\theta)^2 + \sin^2\theta}$$

$$= \frac{9 - 3\cos\theta + 3i\sin\theta}{9 - 6\cos\theta + \cos^2\theta + \sin^2\theta}$$

$$= \frac{9 - 3\cos\theta + 3i\sin\theta}{10 - 6\cos\theta}$$

Equating Re and Im parts

$$C = \frac{9 - 3\cos\theta}{10 - 6\cos\theta} \qquad S = \frac{3\sin\theta}{10 - 6\cos\theta}$$

5) $P = 1 + \cos\theta + \cos 2\theta + \dots + \cos 12\theta \qquad 0 < \theta < \pi$

$Q = \sin\theta + \sin 2\theta + \dots + \sin 12\theta$

a) $P + iQ = 1 + e^{i\theta} + e^{i2\theta} + \dots + e^{i12\theta}$

GP $a = 1$, $r = e^{i\theta}$, terms $= 13$

$$P + iQ = \frac{a(1 - r^n)}{1 - r} = \frac{1(1 - e^{13i\theta})}{1 - e^{i\theta}}$$

$$P + iQ = \frac{e^{-\frac{i\theta}{2}}(1 - e^{13i\theta})}{e^{-i\theta/2} - e^{\frac{i\theta}{2}}}$$

$$= \frac{e^{-\frac{i\theta}{2}} e^{\frac{13i\theta}{2}}\left(e^{-\frac{i13\theta}{2}} - e^{\frac{i13\theta}{2}}\right)}{e^{-i\theta/2} - e^{i\theta/2}}$$

$$P + iQ = \frac{e^{6i\theta}\left(e^{\frac{i13\theta}{2}} - e^{-\frac{i13\theta}{2}}\right)}{e^{i\theta/2} - e^{-\frac{i\theta}{2}}}$$

$$P + iQ = \frac{e^{i6\theta}\left(2i\sin\left(\frac{13\theta}{2}\right)\right)}{2i\sin\left(\frac{\theta}{2}\right)}$$

$$= \frac{e^{i6\theta}\sin\left(\frac{13\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)}$$

$$= (\cos 6\theta + i\sin 6\theta)\sin\left(\frac{13\theta}{2}\right)\operatorname{cosec}\left(\frac{\theta}{2}\right)$$

Equate Re and Im parts

$$P = \cos 6\theta \sin\left(\frac{13\theta}{2}\right)\operatorname{cosec}\left(\frac{\theta}{2}\right)$$

$$Q = \sin 6\theta \sin\left(\frac{13\theta}{2}\right)\operatorname{cosec}\left(\frac{\theta}{2}\right)$$

5 cont) For $P + iQ$ real $Q = 0$ $\qquad 0 < \theta < \pi$

$$\Rightarrow \frac{\sin 6\theta \sin\left(\frac{13\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)} = 0$$

$6\theta = \pi, 2\pi, 3\pi, 4\pi, 5\pi$ $\qquad \theta = \frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{5\pi}{6}$

$\frac{13\theta}{2} = \pi, 2\pi, 3\pi, 4\pi, 5\pi, 6\pi$

$\theta = \frac{2\pi}{13}, \frac{4\pi}{13}, \frac{6\pi}{13}, \frac{8\pi}{13}, \frac{10\pi}{13}, \frac{12\pi}{13}$

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7) $(2+e^{i\theta})(2+e^{-i\theta})$

a) $= 4 + 2e^{i\theta} + 2e^{-i\theta} + 1$

$= 5 + 2(e^{i\theta} + e^{-i\theta}) \qquad = 5 + 4\cos\theta$

b)

$C = 1 - \frac{1}{2}\cos\theta + \frac{1}{4}\cos 2\theta - \frac{1}{8}\cos 3\theta + \dots$

$S = \frac{1}{2}\sin\theta - \frac{1}{4}\sin 2\theta + \frac{1}{8}\sin 3\theta - \dots$

$C - iS = 1 - \frac{1}{2}e^{i\theta} + \frac{1}{4}e^{i2\theta} - \frac{1}{8}e^{i3\theta} + \dots$

GP $a = 1$, $r = -\frac{1}{2}e^{i\theta}$

$C - iS = S_\infty$ of GP $= \frac{a}{1-r} = \frac{1}{1+\frac{1}{2}e^{i\theta}}$

$C - iS = \frac{2}{2+e^{i\theta}} = \frac{2}{2+e^{i\theta}} \times \frac{2+e^{-i\theta}}{2+e^{i\theta}}$

$C - iS = \frac{4+2e^{-i\theta}}{5+4\cos\theta} = \frac{4+2\cos\theta - 2i\sin\theta}{5+4\cos\theta}$

Eq Re and Im

$C = \frac{4+2\cos\theta}{5+4\cos\theta} \qquad S = \frac{2\sin\theta}{5+4\cos\theta}$