8 A computer-controlled machine can be programmed to make cuts by entering the equation of the plane of the cut, and to drill holes by entering the equation of the line of the hole.

A 20 cm × 30 cm × 30 cm cuboid is to be cut and drilled. The cuboid is positioned relative to x-, y- and z-axes as shown in Fig. 8.1.

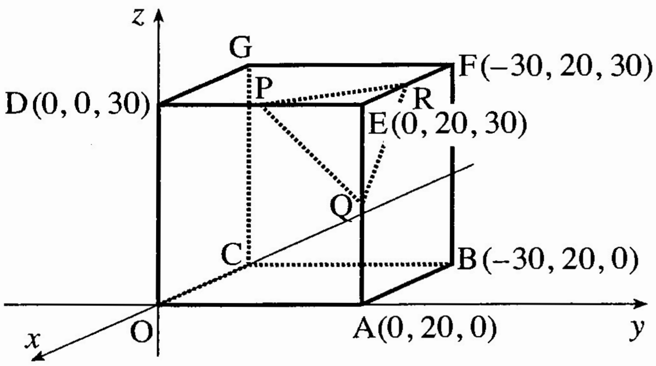

Fig. 8.1

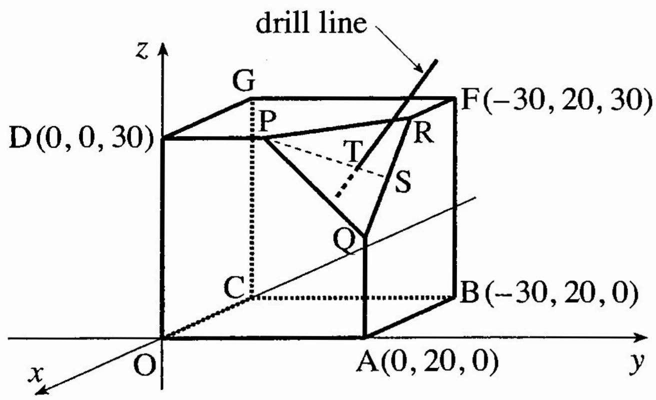

Fig. 8.2

First, a plane cut is made to remove the corner at E. The cut goes through the points P, Q and R, which are the midpoints of the sides ED, EA and EF respectively.

(i) Write down the coordinates of P, Q and R.

Hence show that $\overrightarrow{PQ} = \begin{pmatrix} 0 \\ 10 \\ -15 \end{pmatrix}$ and $\overrightarrow{PR} = \begin{pmatrix} -15 \\ 10 \\ 0 \end{pmatrix}$. [4]

(ii) Show that the vector $\begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix}$ is perpendicular to the plane through P, Q and R.

Hence find the cartesian equation of this plane. [5]

A hole is then drilled perpendicular to triangle PQR, as shown in Fig. 8.2. The hole passes through the triangle at the point T which divides the line PS in the ratio 2:1, where S is the midpoint of QR.

(iii) Write down the coordinates of S, and show that the point T has coordinates $(-5, 16\frac{2}{3}, 25)$. [4]

(iv) Write down a vector equation of the line of the drill hole.

Hence determine whether or not this line passes through C. [5]

3 A triangle ABC has vertices $A(-2, 4, 1)$, $B(2, 3, 4)$ and $C(4, 8, 3)$. By calculating a suitable scalar product, show that angle ABC is a right angle. Hence calculate the area of the triangle. [6]

5 **(i)** Find the cartesian equation of the plane through the point $(2, -1, 4)$ with normal vector

$$\mathbf{n} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}.$$ [3]

(ii) Find the coordinates of the point of intersection of this plane and the straight line with equation

$$\mathbf{r} = \begin{pmatrix} 7 \\ 12 \\ 9 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}.$$ [4]

7

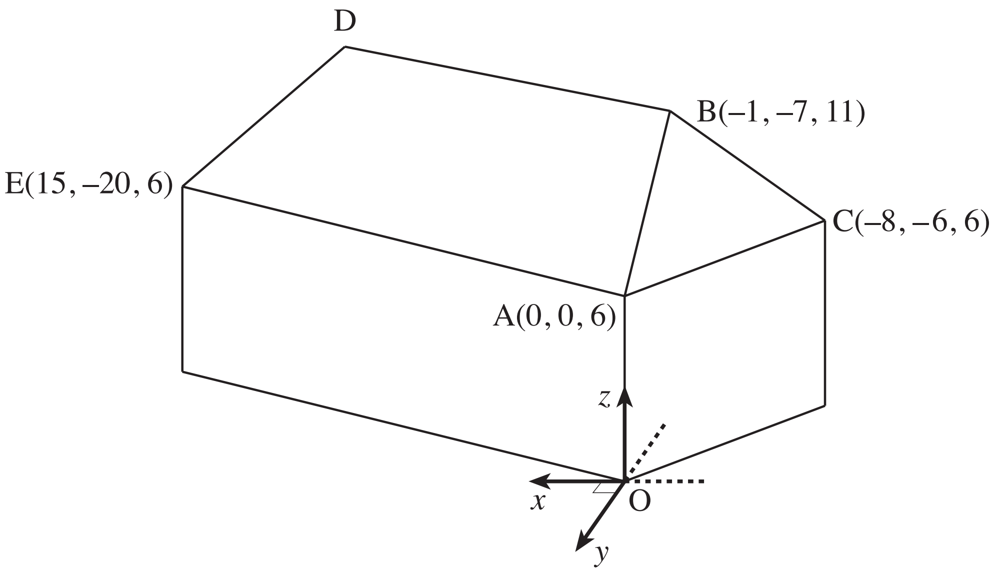

Fig. 7

Fig. 7 illustrates a house. All units are in metres. The coordinates of A, B, C and E are as shown. BD is horizontal and parallel to AE.

(i) Find the length AE. [2]

(ii) Find a vector equation of the line BD. Given that the length of BD is 15 metres, find the coordinates of D. [4]

(iii) Verify that the equation of the plane ABC is

$$-3x + 4y + 5z = 30.$$

Write down a vector normal to this plane. [4]

(iv) Show that the vector $\begin{pmatrix} 4 \\ 3 \\ 5 \end{pmatrix}$ is normal to the plane ABDE. Hence find the equation of the plane ABDE. [4]

(v) Find the angle between the planes ABC and ABDE. [4]

8 A pipeline is to be drilled under a river (see Fig. 8). With respect to axes Oxyz, with the x-axis pointing East, the y-axis North and the z-axis vertical, the pipeline is to consist of a straight section AB from the point A$(0, -40, 0)$ to the point B$(40, 0, -20)$ directly under the river, and another straight section BC. All lengths are in metres.

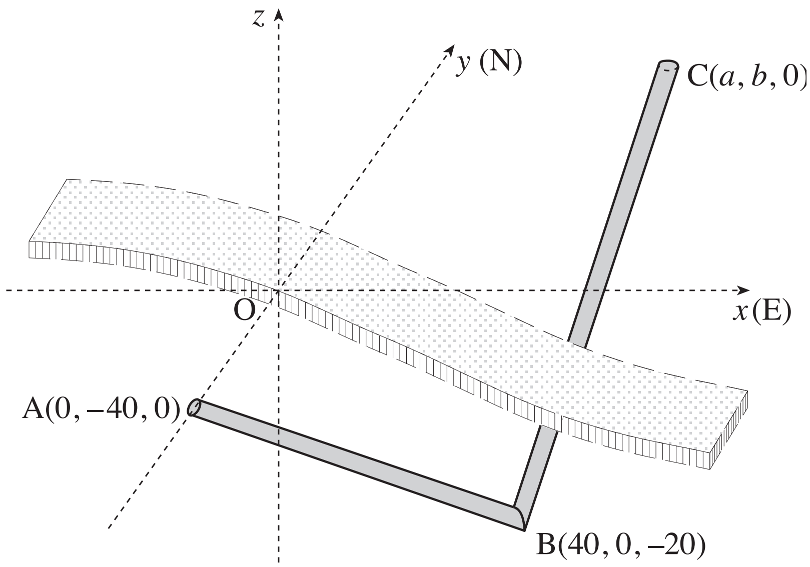

Fig. 8

(i) Calculate the distance AB. [2]

The section BC is to be drilled in the direction of the vector $3\mathbf{i} + 4\mathbf{j} + \mathbf{k}$.

(ii) Find the angle ABC between the sections AB and BC. [4]

The section BC reaches ground level at the point C$(a, b, 0)$.

(iii) Write down a vector equation of the line BC. Hence find a and b. [5]

(iv) Show that the vector $6\mathbf{i} - 5\mathbf{j} + 2\mathbf{k}$ is perpendicular to the plane ABC. Hence find the cartesian equation of this plane. [5]

2 Write down normal vectors to the planes $2x + 3y + 4z = 10$ and $x - 2y + z = 5$.

Hence show that these planes are perpendicular to each other. [4]

5 Verify that the point $(-1, 6, 5)$ lies on both the lines

$$\mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} \quad \text{and} \quad \mathbf{r} = \begin{pmatrix} 0 \\ 6 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}.$$

Find the acute angle between the lines. [7]

7 A glass ornament OABCDEFG is a truncated pyramid on a rectangular base (see Fig. 7). All dimensions are in centimetres.

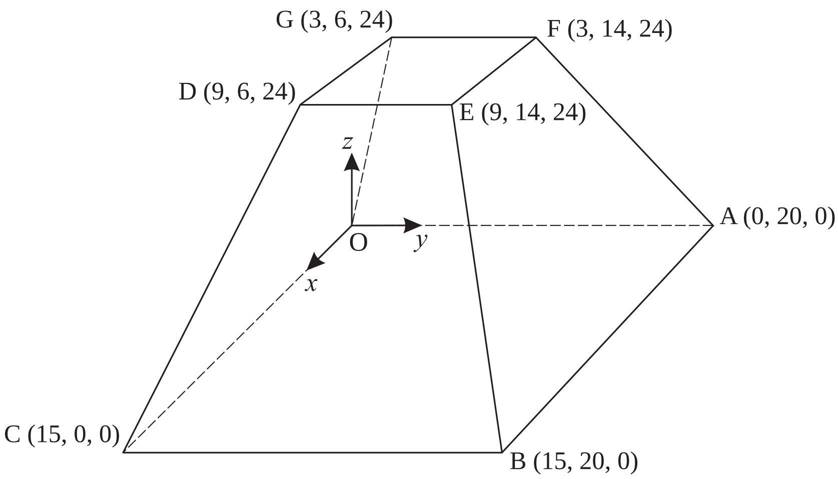

Fig. 7

(i) Write down the vectors $\overrightarrow{CD}$ and $\overrightarrow{CB}$. [2]

(ii) Find the length of the edge CD. [2]

(iii) Show that the vector $4\mathbf{i} + \mathbf{k}$ is perpendicular to the vectors $\overrightarrow{CD}$ and $\overrightarrow{CB}$. Hence find the cartesian equation of the plane BCDE. [5]

(iv) Write down vector equations for the lines OG and AF.

Show that they meet at the point P with coordinates $(5, 10, 40)$. [5]

You may assume that the lines CD and BE also meet at the point P.

The volume of a pyramid is $\frac{1}{3} \times$ area of base $\times$ height.

(v) Find the volumes of the pyramids POABC and PDEFG.

Hence find the volume of the ornament. [4]

8 The upper and lower surfaces of a coal seam are modelled as planes ABC and DEF, as shown in Fig. 8. All dimensions are metres.

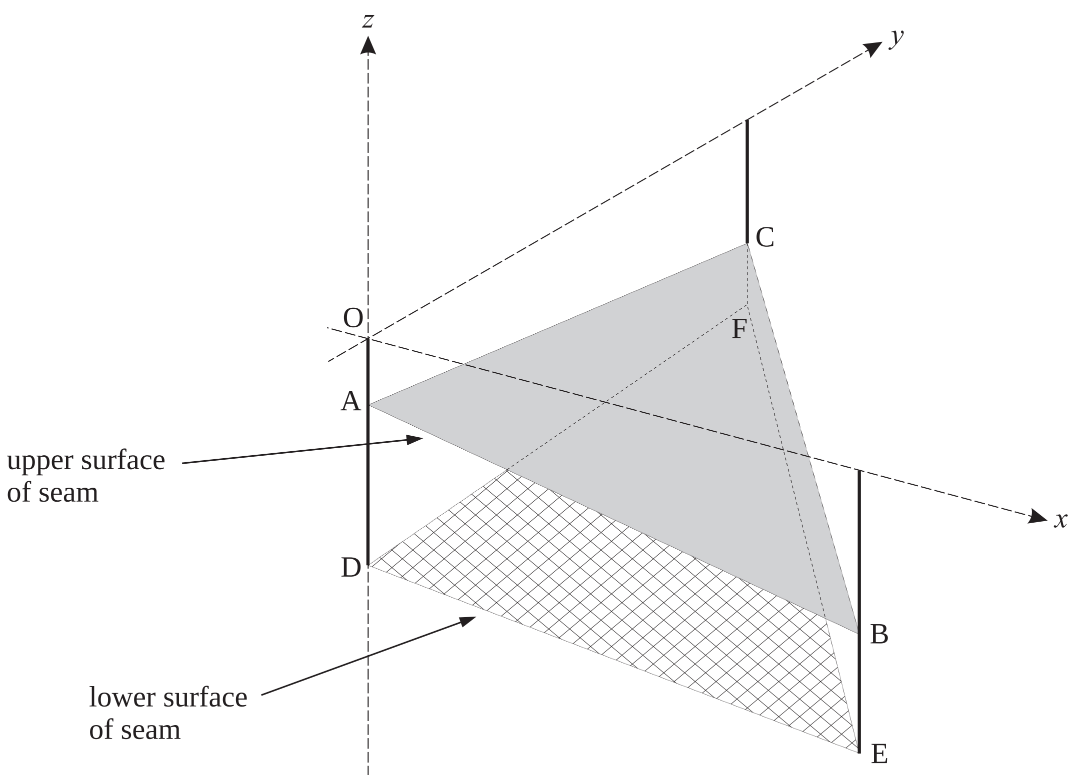

Fig. 8

Relative to axes Ox (due east), Oy (due north) and Oz (vertically upwards), the coordinates of the points are as follows.

A: (0, 0, −15)	B: (100, 0, −30)	C: (0, 100, −25)
D: (0, 0, −40)	E: (100, 0, −50)	F: (0, 100, −35)

(i) Verify that the cartesian equation of the plane ABC is $3x + 2y + 20z + 300 = 0$. [3]

(ii) Find the vectors $\overrightarrow{DE}$ and $\overrightarrow{DF}$. Show that the vector $2\mathbf{i} - \mathbf{j} + 20\mathbf{k}$ is perpendicular to each of these vectors. Hence find the cartesian equation of the plane DEF. [6]

(iii) By calculating the angle between their normal vectors, find the angle between the planes ABC and DEF. [4]

It is decided to drill down to the seam from a point R (15, 34, 0) in a line perpendicular to the upper surface of the seam. This line meets the plane ABC at the point S.

(iv) Write down a vector equation of the line RS.

Calculate the coordinates of S. [5]

3 Vectors **a** and **b** are given by $\mathbf{a} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$ and $\mathbf{b} = 4\mathbf{i} - 2\mathbf{j} + \mathbf{k}$.

Find constants λ and μ such that $\lambda\mathbf{a} + \mu\mathbf{b} = 4\mathbf{j} - 3\mathbf{k}$. **[5]**

5 **(i)** Write down normal vectors to the planes $2x - y + z = 2$ and $x - z = 1$.

Hence find the acute angle between the planes. **[4]**

(ii) Write down a vector equation of the line through $(2,\ 0,\ 1)$ perpendicular to the plane $2x - y + z = 2$. Find the point of intersection of this line with the plane. **[4]**

7 When a light ray passes from air to glass, it is deflected through an angle. The light ray ABC starts at point A (1, 2, 2), and enters a glass object at point B (0, 0, 2). The surface of the glass object is a plane with normal vector **n**. Fig. 7 shows a cross-section of the glass object in the plane of the light ray and **n**.

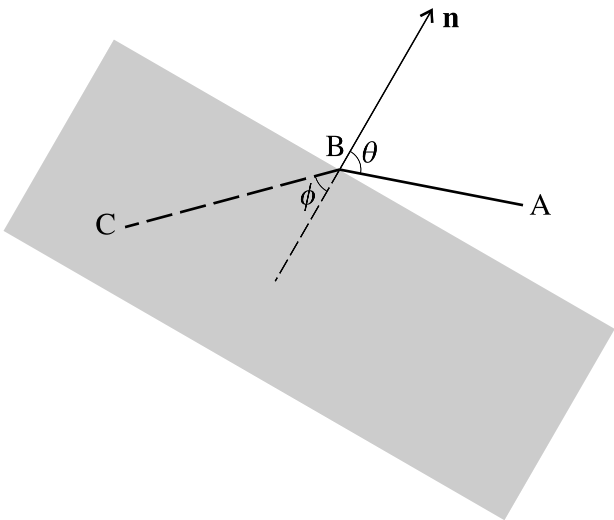

Fig. 7

(i) Find the vector $\overrightarrow{AB}$ and a vector equation of the line AB. **[2]**

The surface of the glass object is a plane with equation $x + z = 2$. AB makes an acute angle θ with the normal to this plane.

(ii) Write down the normal vector **n**, and hence calculate θ, giving your answer in degrees. **[5]**

The line BC has vector equation $\mathbf{r} = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} -2 \\ -2 \\ -1 \end{pmatrix}$. This line makes an acute angle ϕ with the normal to the plane.

(iii) Show that $\phi = 45°$. **[3]**

(iv) Snell's Law states that $\sin\theta = k\sin\phi$, where k is a constant called the refractive index. Find k. **[2]**

The light ray leaves the glass object through a plane with equation $x + z = -1$. Units are centimetres.

(v) Find the point of intersection of the line BC with the plane $x + z = -1$. Hence find the distance the light ray travels through the glass object. **[5]**