MEI CORE 4 JUNE 2005

6) i) $y^2 - x^2 = 4$

Verify $x = t - \frac{1}{t}$, $y = t + \frac{1}{t}$ are parametric eqns of curve

Substituting

$$\left(t+\frac{1}{t}\right)^2 - \left(t-\frac{1}{t}\right)^2$$

$$= t^2 + 2 + \frac{1}{t^2} - \left(t^2 - 2 + \frac{1}{t^2}\right)$$

$$= t^2 + 2 + \frac{1}{t^2} - t^2 + 2 - \frac{1}{t^2}$$

$$= 4 \quad \text{as required}$$

6ii) $\frac{dy}{dt} = 1 - \frac{1}{t^2}$, $\frac{dx}{dt} = 1 + \frac{1}{t^2}$

$$\frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt}$$

$$= \frac{1 - \frac{1}{t^2}}{1 + \frac{1}{t^2}}$$

$$= \frac{\frac{t^2-1}{t^2}}{\frac{t^2+1}{t^2}}$$

$$= \frac{t^2 - 1}{t^2+1}$$

$$= \frac{(t-1)(t+1)}{t^2+1}$$

At st pt $\frac{dy}{dx} = 0$

$$\Rightarrow (t-1)(t+1) = 0$$

$$\Rightarrow t = 1 \text{ or } t = -1$$

when $t = 1$, $y = 1 + \frac{1}{1} = 2$

$x = 1 - \frac{1}{1} = 0$

$\therefore (0, 2)$ a stationary point

when $t = -1$, $y = -1 + \frac{1}{-1} = -2$

$x = -1 - \frac{1}{-1} = 0$

$\therefore (0, -2)$ a stationary point

MEI CORE 4 JAN 2006

2)

$x = t - \ln t$ $(t > 0)$

$y = t + \ln t$

$\frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt}$

$\frac{dy}{dx} = \dfrac{1 + \frac{1}{t}}{1 - \frac{1}{t}}$

When $t = 2$

$\frac{dy}{dx} = \dfrac{1 + \frac{1}{2}}{1 - \frac{1}{2}}$

$= \dfrac{\frac{3}{2}}{\frac{1}{2}} = 3$

6)

$x = a(\theta - \sin\theta), \quad y = a(1 - \cos\theta)$

i)

A) Find OE

At E x has max value

$y = 0$

$y = 0 \Rightarrow 1 - \cos\theta = 0$

$\Rightarrow \cos\theta = 1$

$\Rightarrow \theta = 0$ or 2π

O is point $(0, 0)$

E is point $\left(a(2\pi - \sin 2\pi), 0\right)$

E is $(2a\pi, 0)$

$\therefore$ OE $= 2a\pi - 0 = 2a\pi$

B)

Max $y = a(1 - \cos\theta)$

when $\cos\theta = -1$

Max height $= a(1 - -1) = 2a$

ii)

$$\frac{dy}{d\theta} = +a\sin\theta$$

$$\frac{dx}{d\theta} = a(1 - \cos\theta)$$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \Big/ \frac{dx}{d\theta}$$

$$= \frac{a\sin\theta}{a(1-\cos\theta)} = \frac{\sin\theta}{1-\cos\theta}$$

iii)

At B, AB is a tgt

Grad of AB $= \tan^{-1} 30^\circ = \frac{1}{\sqrt{3}}$

$\therefore$ gradient of curve at B $= \frac{1}{\sqrt{3}}$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{3}}$$

$$\Rightarrow \frac{\sin\theta}{1-\cos\theta} = \frac{1}{\sqrt{3}}$$

$$\Rightarrow \sin\theta = \frac{1}{\sqrt{3}}(1 - \cos\theta)$$

6 iii) cont)

$$\sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}$$

$$\frac{1}{\sqrt{3}}\left(1-\cos\frac{2\pi}{3}\right) = \frac{1}{\sqrt{3}}\left(1 - -\frac{1}{2}\right)$$

$$= \frac{1}{\sqrt{3}} \times \frac{3}{2} = \frac{\sqrt{3}}{2}$$

$\therefore\ \theta = \frac{2\pi}{3}$ is a solution of

$$\sin\theta = \frac{1}{\sqrt{3}}\left(1-\cos\theta\right)$$

BF is y coord of B

$$y = a\left(1-\cos\theta\right)$$

at B, $\theta = \frac{2\pi}{3}$ so

$$y = a\left(1-\cos\frac{2\pi}{3}\right)$$

$$= a\left(1 - -\frac{1}{2}\right) = \frac{3a}{2}$$

$$\Rightarrow BF = \frac{3a}{2}$$

OF is same as x coord of B

$$x = a\left(\theta - \sin\theta\right)$$

$$x = a\left(\frac{2\pi}{3} - \sin\frac{2\pi}{3}\right)$$

$$x = a\left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right)$$

$$\therefore\ OF = a\left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right)$$

6iv) Due to symmetry

$$BC = OE - 2 \times OF$$

$$= 2a\pi - 2\left(a\left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right)\right)$$

$$= 2a\pi - \frac{4a\pi}{3} + \sqrt{3}a$$

$$= \frac{2a\pi}{3} + \sqrt{3}a$$

$$BC = a\left(\frac{2\pi}{3} + \sqrt{3}\right)$$

$$\tan 30^\circ = \frac{1}{\sqrt{3}} = \frac{BF}{AF}$$

$$\Rightarrow AF = \sqrt{3}\,BF$$

$$AF = \sqrt{3} \times \frac{3a}{2}$$

$$AF = \frac{3\sqrt{3}a}{2}$$

Due to symmetry

$$AD = 2AF + BC$$

$$= 2\left(\frac{3\sqrt{3}a}{2}\right) + a\left(\frac{2\pi}{3} + \sqrt{3}\right)$$

$$\therefore\ 20 = 3\sqrt{3}a + a\left(\frac{2\pi}{3} + \sqrt{3}\right)$$

$$20 = a\left(4\sqrt{3} + \frac{2\pi}{3}\right)$$

$$a = \frac{20}{\left(4\sqrt{3} + \frac{2\pi}{3}\right)} = 2.22\text{ m} \quad \text{to 3 s.f.}$$

Section B

7) i) $x = \cos\theta \quad y = \sin\theta - \frac{1}{8}\sin 2\theta$

At A, $x = 1$

$\Rightarrow \cos\theta = 1$

$\Rightarrow \theta = 0$

At B, $x = -1$

$\Rightarrow \cos\theta = -1$

$\Rightarrow \theta = \pi$

At C, $x = 0$

$\Rightarrow \cos\theta = 0$

$\Rightarrow \theta = \frac{\pi}{2}$ or $\frac{3\pi}{2}$

when $\theta = \frac{\pi}{2}$, $y = 1 - 0 = 1$

when $\theta = \frac{3\pi}{2}$, $y = -1 - 0 = -1$

$\therefore$ C is point $(0, 1)$

7ii)

$$\frac{dy}{d\theta} = \cos\theta - \frac{1}{4}\cos 2\theta$$

$$\frac{dx}{d\theta} = -\sin\theta$$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \Big/ \frac{dx}{d\theta}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\cos\theta - \frac{1}{4}\cos 2\theta}{-\sin\theta}$$

$$\frac{dy}{dx} = \frac{4\cos\theta - \cos 2\theta}{-4\sin\theta}$$

$$\frac{dy}{dx} = \frac{\cos 2\theta - 4\cos\theta}{4\sin\theta}$$

At max pt $\frac{dy}{dx} = 0$

$\Rightarrow \cos 2\theta - 4\cos\theta = 0$

$\Rightarrow 2\cos^2\theta - 1 - 4\cos\theta = 0$

7iii)

$2\cos^2\theta - 4\cos\theta - 1 = 0$

$$\cos\theta = \frac{4 \pm \sqrt{16 + 8}}{4}$$

$\cos\theta = \cancel{2.225}$ or -0.2247

$\Rightarrow \theta = 1.7975$ radians

$\Rightarrow y = \sin(1.7975) - \frac{1}{8}\sin(2 \times 1.7975)$

$y = 1.029$ to 4 s.f.

7iv)

$$y = \frac{1}{4}(4 - x)\sqrt{1 - x^2}$$

$$\text{Volume} = \int_{-1}^{1} \pi y^2 \, dx$$

$$= \pi \int_{-1}^{1} \frac{1}{16}(4 - x)^2 (1 - x^2)\, dx$$

$$= \frac{\pi}{16}\int_{-1}^{1} (16 - 8x + x^2)(1 - x^2)\, dx$$

7iv) cont

$$= \frac{\pi}{16}\int_{-1}^{1}\left(16-8x+x^2-16x^2+8x^3-x^4\right)dx$$

$$= \frac{\pi}{16}\int_{-1}^{1}\left(16-8x-15x^2+8x^3-x^4\right)dx$$

$$= \frac{\pi}{16}\left[16x-4x^2-5x^3+2x^4-\frac{x^5}{5}\right]_{-1}^{1}$$

$$= \frac{\pi}{16}\left[\left(16-4-5+2-\frac{1}{5}\right) - \left(-16-4+5+2+\frac{1}{5}\right)\right]$$

$$= \frac{\pi}{16}\left[32-10-\frac{2}{5}\right]$$

$$= \frac{27\pi}{20} = 4.24$$

MEI CORE 4 JUNE 2007

8) $x = 10\cos\theta + 5\cos 2\theta$
$y = 10\sin\theta + 5\sin 2\theta$
$(0 \le \theta < 2\pi)$

i) $\frac{dy}{d\theta} = 10\cos\theta + 10\cos 2\theta$

$\frac{dx}{d\theta} = -10\sin\theta - 10\sin 2\theta$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{10\cos\theta + 10\cos 2\theta}{-10\sin\theta - 10\sin 2\theta}$$

$$\frac{dy}{dx} = -\frac{\cos\theta + \cos 2\theta}{\sin\theta + \sin 2\theta}$$

when $\theta = \frac{\pi}{3}$

$$\frac{dy}{dx} = -\frac{\cos\frac{\pi}{3} + \cos\frac{2\pi}{3}}{\sin\frac{\pi}{3} + \sin\frac{2\pi}{3}}$$

$$= -\left(\frac{\frac{1}{2} + -\frac{1}{2}}{\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2}}\right)$$

$$= 0$$

At A, $\frac{dy}{dx} = 0$

$x = 10\cos\frac{\pi}{3} + 5\cos\frac{2\pi}{3}$

$x = 10 \times \frac{1}{2} + 5(-\frac{1}{2}) = 5/2$

$y = 10\sin\frac{\pi}{3} + 5\sin\frac{2\pi}{3}$

$y = 10 \times \frac{\sqrt{3}}{2} + 5 \times \frac{\sqrt{3}}{2} = \frac{15\sqrt{3}}{2}$

A is point $\left(\frac{5}{2}, \frac{15\sqrt{3}}{2}\right)$

8ii) $x^2 + y^2 =$

$(10\cos\theta + 5\cos 2\theta)^2 + (10\sin\theta + 5\sin 2\theta)^2$

$= 100\cos^2\theta + 100\cos\theta\cos 2\theta + 25\cos^2 2\theta$
$+ 100\sin^2\theta + 100\sin\theta\sin 2\theta + 25\sin^2 2\theta$

$= 100(\cos^2\theta + \sin^2\theta) + 25(\cos^2 2\theta + \sin^2 2\theta)$
$+ 100(\cos\theta\cos 2\theta + \sin\theta\sin 2\theta)$

$= 100 + 25 + 100\cos(2\theta - \theta)$

$= 125 + 100\cos\theta$

8iii) $(\text{Distance from } O)^2 = x^2 + y^2$

Max Dist^2 $= 125 + 100 = 225$

Max Distance $= 15$ m

Min Dist^2 $= 125 - 100 = 25$

Min Distance $= 5$ m

8iv) $2\cos^2\theta + 2\cos\theta - 1 = 0$

By calculator $\cos\theta = 0.3660254$
$\cos\theta = -1.3660254$

Now $x^2 + y^2 = 125 + 100\cos\theta$
and at B, $x = 0$

$\therefore y^2 = 125 + 100 \times 0.3660254$

$\Rightarrow y^2 = 161.60254$

$\Rightarrow y = 12.712$

$\therefore OB = 12.7$ m to 3 s.f.

//

8) i) $x = k\cos\theta \, , \quad y = \frac{1}{2}k\sin\theta$

$\Rightarrow \quad \frac{x}{k} = \cos\theta \, , \quad \frac{2y}{k} = \sin\theta$

$\Rightarrow \quad \frac{x^2}{k^2} + \frac{4y^2}{k^2} = \cos^2\theta + \sin^2\theta = 1$

$\Rightarrow \quad x^2 + 4y^2 = k^2$

$\therefore$ $x = k\cos\theta$ and $y = \frac{1}{2}k\sin\theta$ are parametric equns for the given curve

8ii)

$$x^2 + 4y^2 = k^2$$

Differentiate implicitly

$$2x + 8y\frac{dy}{dx} = 0$$

$$8y\frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = -\frac{2x}{8y} = -\frac{x}{4y}$$

8ii) alternative method

$$\frac{dy}{d\theta} = \frac{1}{2}k\cos\theta$$

$$\frac{dx}{d\theta} = -k\sin\theta$$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \Big/ \frac{dx}{d\theta} = \frac{\frac{1}{2}k\cos\theta}{-k\sin\theta}$$

$$= \frac{\frac{1}{2}x}{-k \times \frac{2y}{k}} = -\frac{x}{4y}$$

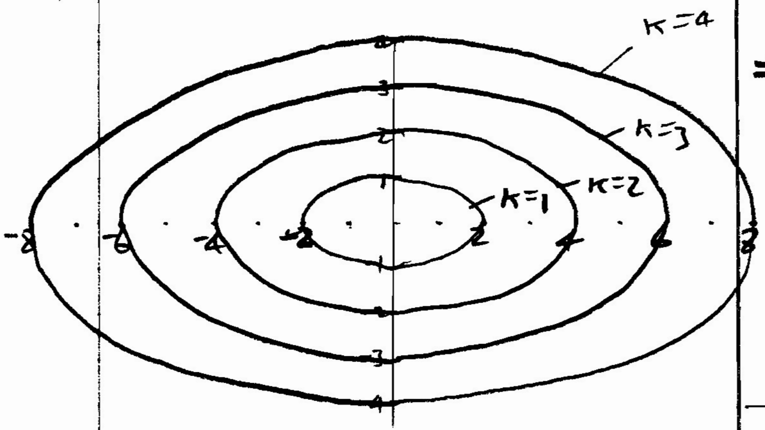
MEI CORE 4 JANUARY 2008

8iii) Point $(2,0)$ on curve

$$2^2 + 4(0^2) = k^2$$

$$\Rightarrow k = 2$$

8iv)



8v)

If path of stream $\perp$ to contour

Gradient of stream path on map $\times$ gradient of contour $= -1$

$$\therefore \frac{dy}{dx}_{\text{stream}} = -\frac{1}{\frac{dy}{dx}_{\text{contour}}}$$

$$= -\frac{1}{\frac{-x}{4y}} = \frac{4y}{x}$$

8vi)

$$\frac{dy}{dx} = \frac{4y}{x}$$

$$\int \frac{1}{y}\,dy = \int \frac{4}{x}\,dx$$

$$\ln y = 4\ln x + c$$

Given $(2,1)$ on path

$$\ln(1) = 4\ln(2) + c$$

$$\Rightarrow c = -4\ln 2$$

$$\therefore \ln y = 4\ln x - 4\ln 2$$

$$\Rightarrow \ln y = 4\ln\left(\frac{x}{2}\right)$$

$$\Rightarrow \ln y = \ln\left(\frac{x}{2}\right)^4$$

$$\Rightarrow y = \left(\frac{x}{2}\right)^4$$

$$\Rightarrow y = \frac{x^4}{16}$$

5)

$x = 1 + u^2 \quad , \quad y = 2u^3$

i)

$\frac{dx}{du} = 2u \quad , \quad \frac{dy}{du} = 6u^2$

$\frac{dy}{dx} = \frac{dy}{du} \Big/ \frac{dx}{du} = \frac{6u^2}{2u} = 3u$

$\frac{dy}{dx} = 3u$

ii) At $(5, 16)$ $\quad u = 2$

$\therefore$ gradient $= 6$

MEI CORE 4 JAN 2009

8) $x = 2 + 2\sin\theta$

$y = 2\cos\theta + \sin 2\theta$

$(0 \le \theta \le 2\pi)$

i) $\frac{dy}{d\theta} = -2\sin\theta + 2\cos 2\theta$

$\frac{dx}{d\theta} = 2\cos\theta$

$\frac{dy}{dx} = \frac{dy}{d\theta} \Big/ \frac{dx}{d\theta}$

$\frac{dy}{dx} = \frac{2\cos 2\theta - 2\sin\theta}{2\cos\theta}$

$\frac{dy}{dx} = \frac{\cos 2\theta - \sin\theta}{\cos\theta}$

8 ii) When $\theta = \frac{\pi}{6}$

$\frac{dy}{dx} = \frac{\cos\frac{\pi}{3} - \sin\frac{\pi}{6}}{\cos\frac{\pi}{6}}$

$= \frac{\frac{1}{2} - \frac{1}{2}}{\frac{\sqrt{3}}{2}} = 0$

When $\theta = \frac{\pi}{6}$

$y = 2\cos\frac{\pi}{6} + \sin\frac{\pi}{3} = \sqrt{3} + \frac{\sqrt{3}}{2}$

$= \frac{3\sqrt{3}}{2}$

$x = 2 + 2\sin\frac{\pi}{6} = 3$

B is point $\left(3, \frac{3\sqrt{3}}{2}\right)$

Assuming curve is symmetrical about x-axis

$|BC| = 2 \times \frac{3\sqrt{3}}{2} = 3\sqrt{3}$ m

8iii)

A) $x\cos\theta = 2\cos\theta + 2\sin\theta\cos\theta$
$= 2\cos\theta + \sin 2\theta = y$

B) $2\sin\theta = x - 2$
$\sin\theta = \frac{x-2}{2}$

$\Rightarrow \cos^2\theta = (1 - \sin^2\theta) = 1 - \left(\frac{x-2}{2}\right)^2$

$\Rightarrow \cos^2\theta = 1 - \frac{(x^2 - 4x + 4)}{4}$

$= x - \frac{x^2}{4}$

c) $y^2 = x^2\cos^2\theta = x^2\left(x - \frac{x^2}{4}\right)$

$\Rightarrow y^2 = x^3 - \frac{x^4}{4}$

iv) $V = \pi\int_0^4 y^2\,dx$

$V = \pi\int_0^4 \left(x^3 - \frac{x^4}{4}\right) dx$

$V = \pi\left[\frac{x^4}{4} - \frac{x^5}{20}\right]_0^4$

$V = \pi\left[(64 - 51.2) - (0 - 0)\right]$

$V = 12.8\pi$ or 40.2 m^3

5) $x = at^3 \qquad y = \dfrac{a}{1+t^2}$

$\dfrac{dx}{dt} = 3at^2 \qquad \dfrac{dy}{dt} = \dfrac{(1+t^2)0 - a(2t)}{(1+t^2)^2}$

$\dfrac{dy}{dt} = \dfrac{-2at}{(1+t^2)^2}$

$\dfrac{dy}{dx} = \dfrac{dy}{dt} \Big/ \dfrac{dx}{dt}$

$\dfrac{dy}{dx} = \dfrac{\dfrac{-2at}{(1+t^2)^2}}{3at^2}$

$\dfrac{dy}{dx} = \dfrac{-2}{3t(1+t^2)^2}$

At point $(a, \frac{1}{2}a) \qquad t = 1$

$\dfrac{dy}{dx} = \dfrac{-2}{3 \times 1 \times 2^2}$

$\dfrac{dy}{dx} = -\dfrac{1}{6}$