

Integration by Substitution

- (iii) Using the substitution $u = 2 - \cos x$ or otherwise, find the exact value of $\int_0^\pi \frac{\sin x}{2 - \cos x} dx$. [4]

$$\begin{aligned}&= \int_1^3 \frac{1}{u} du \\&= [\ln u]_1^3 \\&= \ln 3 - \ln 1 \\&= \ln 3\end{aligned}$$

Let $u = 2 - \cos x$

$$\frac{du}{dx} = \sin x$$

$$du = \sin x dx$$

Limits

$$x = \pi \quad u = 3$$

$$x = 0 \quad u = 1$$

- 7 A curve is defined by the equation $y = 2x \ln(1+x)$.

- (i) Find $\frac{dy}{dx}$ and hence verify that the origin is a stationary point of the curve. [4]

- (ii) Find $\frac{d^2y}{dx^2}$, and use this to verify that the origin is a minimum point. [5]

- (iii) Using the substitution $u = 1+x$, show that $\int \frac{x^2}{1+x} dx = \int \left(u - 2 + \frac{1}{u}\right) du$.

Hence evaluate $\int_0^1 \frac{x^2}{1+x} dx$, giving your answer in an exact form. [6]

∴ $y = 2x \ln(1+x)$

$$\frac{dy}{dx} = 2x \times \frac{1}{1+x} + 2 \ln(1+x)$$

$$\frac{d^2y}{dx^2} = \frac{2x}{(1+x)^2} + 2 \ln(1+x)$$

At origin $x = 0$

$$\frac{dy}{dx} = \frac{2(0)}{1+0} + 2\ln(1+0) = 0 + 0 = 0$$

\therefore st pt at origin

ii) $\frac{dy}{dx} = \frac{2x}{1+x} + 2\ln(1+x)$

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{(1+x)x2 - 2x(1)}{(1+x)^2} + \frac{2}{1+x} \\ &= \frac{2 + 2x - 2x}{(1+x)^2} + \frac{2}{1+x}\end{aligned}$$

$$\frac{d^2y}{dx^2} = \frac{2}{(1+x)^2} + \frac{2}{1+x}$$

At $(0, 0)$ $\frac{d^2y}{dx^2} = \frac{2}{1} + \frac{2}{1} = 4 > 0$

\therefore a minimum

(iii) Using the substitution $u = 1 + x$, show that $\int \frac{x^2}{1+x} dx = \int \left(u - 2 + \frac{1}{u}\right) du$.

Hence evaluate $\int_0^1 \frac{x^2}{1+x} dx$, giving your answer in an exact form.

$$\int \frac{x^2}{1+x} dx$$

Let $v = 1+x$
 $\frac{dv}{dx} = 1$
 $dv = dx$

$$= \int \frac{(v-1)^2}{v} dv$$

$$= \int \frac{v^2 - 2v + 1}{v} dv$$

$$= \int \left(v - 2 + \frac{1}{v}\right) dv$$

Limits

$$x = 1 \quad v = 2$$

$$x = 0 \quad v = 1$$

$$\int_0^1 \frac{x^2}{1+x} dx = \int_1^2 \left(v - 2 + \frac{1}{v}\right) dv$$

$$= \left[\frac{v^2}{2} - 2v + \ln v \right]_1^2$$

$$= \left(\frac{4}{2} - 4 + \ln 2 \right) - \left(\frac{1}{2} - 2 + \ln 1 \right)$$

$$= -2 + \ln 2 + \frac{3}{2}$$

$$= \ln 2 - \frac{1}{2}$$

- 7 Fig. 7 shows part of the curve $y = f(x)$, where $f(x) = x\sqrt{1+x}$. The curve meets the x -axis at the origin and at the point P.

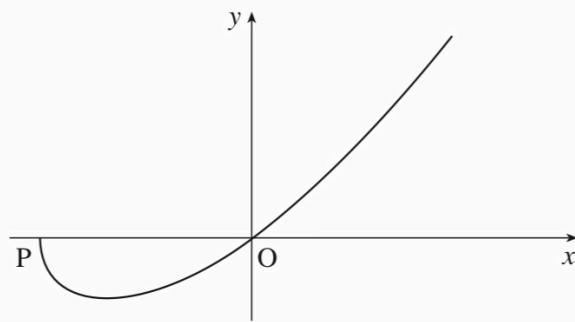


Fig. 7

- (i) Verify that the point P has coordinates $(-1, 0)$. Hence state the domain of the function $f(x)$. [2]
- (ii) Show that $\frac{dy}{dx} = \frac{2+3x}{2\sqrt{1+x}}$. [4]
- (iii) Find the exact coordinates of the turning point of the curve. Hence write down the range of the function. [4]
- (iv) Use the substitution $u = 1+x$ to show that

$$\int_{-1}^0 x\sqrt{1+x} dx = \int_0^1 (u^{\frac{3}{2}} - u^{\frac{1}{2}}) du.$$

Hence find the area of the region enclosed by the curve and the x -axis. [8]

HWK Q7 above for Wed 4th December