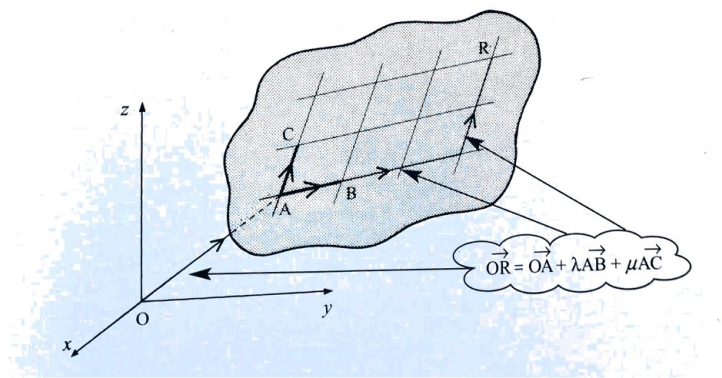
Equation of a plane

To find the vector form of the equation of the plane through the points A, B and C (with position vectors $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$, $\overrightarrow{OC} = \mathbf{c}$), think of starting at the origin, travelling along OA to join the plane at A, and then any distance in each of the directions $\overrightarrow{AB}$ and $\overrightarrow{AC}$ to reach a general point R with position vector $\mathbf{r}$, where

$$\mathbf{r} = \overrightarrow{OA} + \lambda\overrightarrow{AB} + \mu\overrightarrow{AC}.$$



Example in Vector Form

Plane through $A(4,2,0)$
$B(3,1,1)$
$C(4,-1,1)$

$$\underline{r} = \overrightarrow{OA} + \lambda\overrightarrow{AB} + \mu\overrightarrow{AC}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \underline{r} = \begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix} + \lambda\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} + \mu\begin{pmatrix} 0 \\ -3 \\ 1 \end{pmatrix}$$

Cartesian Form

$$x = 4 - \lambda \quad (1)$$
$$y = 2 - \lambda - 3\mu \quad (2)$$
$$z = \lambda + \mu \quad (3)$$

① + ③ $\quad x + z = 4 + \mu$

② + ③ $\quad y + z = 2 - 2\mu$

From ① $\quad \mu = x + z - 4$

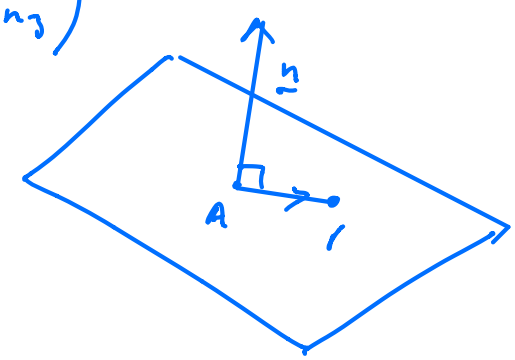
so $\quad y + z = 2 - 2(x + z - 4)$

$$y + z = 2 - 2x - 2z + 8$$

$$\underline{2x + y + 3z = 10}$$

Direction Perpendicular to a Plane

$$\underline{n} = \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} \qquad \overrightarrow{OA} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$



$$\underline{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$(\underline{r} - \underline{a}) \cdot \underline{n} = 0$$

$$\underline{r} \cdot \underline{n} - \underline{a} \cdot \underline{n} = 0$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} = \underline{a} \cdot \underline{n}$$

$$n_1 x + n_2 y + n_3 z = d$$

where $d = \underline{a} \cdot \underline{n}$

Note the coefficients of x, y and z give the components of a normal to the plane

Intersection of a Line and a Plane

Ex Find intersection of

$$\underline{r} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \quad \text{and} \quad 5x + y - z = 1$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 + \lambda \\ 3 + 2\lambda \\ 4 - \lambda \end{pmatrix}$$

Sub in plane

$$5(2+\lambda) + (3+2\lambda) - (4-\lambda) = 1$$

$$10 + 5\lambda + 3 + 2\lambda - 4 + \lambda = 1$$

$$8\lambda = -8$$

$$\underline{\lambda = -1}$$

Sub in line

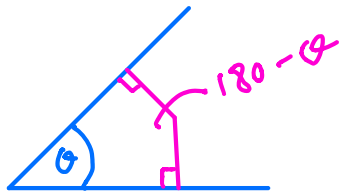
$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 - 1 \\ 3 - 2 \\ 4 + 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix}$$

Point of intersection $(1, 1, 5)$

Perpendicular Distance from a Point (α, β, γ) to plane $n_1 x + n_2 y + n_3 z + d = 0$ is given by

$$\frac{|n_1\alpha + n_2\beta + n_3\gamma + d|}{\sqrt{n_1^2 + n_2^2 + n_3^2}}$$

Angle between two planes



Find the angle between normals to the plane

If obtuse and asked for acute, subtract from 180°

Exercise 11F

Q13

i) $A(4,1,3)$
$B(6,4\ 8)$

$$\underline{r} = \begin{pmatrix} 4 \\ 1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}$$

ii) $$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4+2\lambda \\ 1+3\lambda \\ 3+5\lambda \end{pmatrix}$$

Sub in plane $x + 2y - z + 3 = 0$

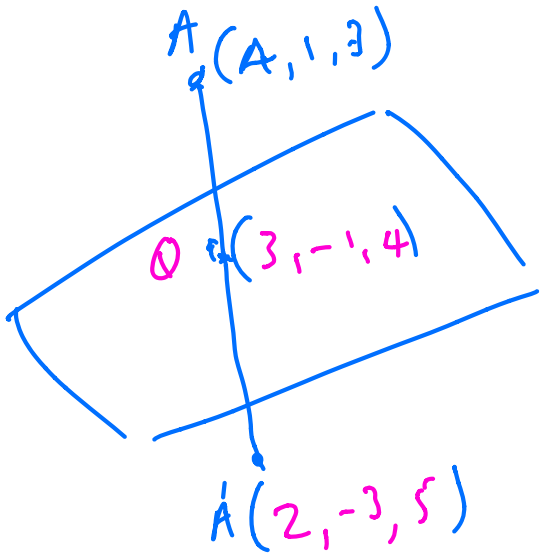
$$4 + 2\lambda + 2(1+3\lambda) - (3+5\lambda) + 3 = 0$$
$$4 + 2\lambda + 2 + 6\lambda - 3 - 5\lambda + 3 = 0$$
$$3\lambda = -6$$
$$\lambda = -2$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 & -4 \\ 1 & -6 \\ 3 & -10 \end{pmatrix} = \begin{pmatrix} 0 \\ -5 \\ -7 \end{pmatrix}$$

Intersect at $P(0, -5, -7)$

iii)

Line is $\underline{r} = \begin{pmatrix} 4 \\ 1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$



Find intersection

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4+\lambda \\ 1+2\lambda \\ 3-\lambda \end{pmatrix}$$

sub in plane

$$4+\lambda+2(1+2\lambda)-(3-\lambda)+3=0$$

$$4+\lambda+2+4\lambda-3+\lambda+3=0$$

$$6\lambda=-6$$

$$\lambda=-1$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4-1 \\ 1-2 \\ 3--1 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix}$$

$Q(3,-1,4)$

$A'(2,-3,5)$

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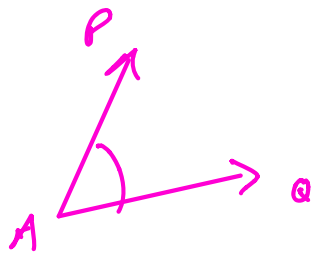
iv) $\angle PAQ$

$A(4,1,3)$
$P(0,-5,-7)$
$Q(3,-1,4)$



$$\cos\angle PAQ = \frac{\vec{AP}\cdot\vec{AQ}}{|\vec{AP}||\vec{AQ}|}$$

$$\vec{AP} = \begin{pmatrix} -4 \\ -6 \\ -10 \end{pmatrix}$$

$$\vec{AQ} = \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}$$

$$\cos\angle PAQ = \frac{4+12-10}{\sqrt{16+36+100}\sqrt{1+4+1}} = \frac{6}{\sqrt{152}\sqrt{6}}$$

$$\theta = 78.5^\circ$$

HWK Exercise 11F Q14