Proof by induction ME 8

1 Let $\text{f}(n) = 9^n - 1$, where $n \in \mathbb{Z}^+$.

$\therefore \text{f}(1) = 9^1 - 1 = 8$, which is divisible by 8.

$\therefore \text{f}(n)$ is divisible by 8 when $n = 1$.

Assume that for $n = k$,

$\text{f}(k) = 9^k - 1$ is divisible by 8 for $k \in \mathbb{Z}^+$.

$$\begin{aligned}\therefore \text{f}(k+1) &= 9^{k+1} - 1 \\ &= 9^k . 9^1 - 1 \\ &= 9(9^k) - 1\end{aligned}$$

$$\begin{aligned}\therefore \text{f}(k+1) - \text{f}(k) &= [9(9^k) - 1] - [9^k - 1] \\ &= 9(9^k) - 1 - 9^k + 1 \\ &= 8(9^k)\end{aligned}$$

$$\therefore \text{f}(k+1) = \text{f}(k) + 8(9^k)$$

As both f (k) and $8(9^k)$ are divisible by 8 then the sum of these two terms must also be divisible by 8. Therefore f (n) is divisible by 8 when $n = k + 1$.

If f (n) is divisible by 8 when $n = k$, then it has been shown that f (n) is also divisible by 8 when $n = k + 1$. As f (n) is divisible by 8 when $n = 1$, f (n) is also divisible by 8 for all $n \geqslant 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.

2 **a** $\mathbf{B}^2 = \mathbf{BB} = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 1+0 & 0+0 \\ 0+0 & 0+9 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 9 \end{pmatrix}$

$\mathbf{B}^3 = \mathbf{B}^2\mathbf{B} = \begin{pmatrix} 1 & 0 \\ 0 & 9 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 1+0 & 0+0 \\ 0+0 & 0+27 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 27 \end{pmatrix}$

b As $\mathbf{B}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 3^2 \end{pmatrix}$ and $\mathbf{B}^3 = \begin{pmatrix} 1 & 0 \\ 0 & 3^3 \end{pmatrix}$, we suggest that $\mathbf{B}^n = \begin{pmatrix} 1 & 0 \\ 0 & 3^n \end{pmatrix}$.

2 c $n=1$; LHS $=\begin{pmatrix}1 & 0\\0 & 3\end{pmatrix}^1=\begin{pmatrix}1 & 0\\0 & 3\end{pmatrix}$

RHS $=\begin{pmatrix}1 & 0\\0 & 3^1\end{pmatrix}=\begin{pmatrix}1 & 0\\0 & 3\end{pmatrix}$

As LHS=RHS, the matrix equation is true for $n = 1$.

Assume that the matrix equation is true for $n = k$.

i.e. $\begin{pmatrix}1 & 0\\0 & 3\end{pmatrix}^k=\begin{pmatrix}1 & 0\\0 & 3^k\end{pmatrix}$

With $n = k + 1$ the matrix equation becomes

$$\begin{pmatrix}1 & 0\\0 & 3\end{pmatrix}^{k+1}=\begin{pmatrix}1 & 0\\0 & 3\end{pmatrix}^k\begin{pmatrix}1 & 0\\0 & 3\end{pmatrix}$$
$$=\begin{pmatrix}1 & 0\\0 & 3^k\end{pmatrix}\begin{pmatrix}1 & 0\\0 & 3\end{pmatrix}$$
$$=\begin{pmatrix}1+0 & 0+0\\0+0 & 0+3(3^k)\end{pmatrix}$$
$$=\begin{pmatrix}1 & 0\\0 & 3^{k+1}\end{pmatrix}$$

Therefore the matrix is true when $n = k+1$.

If the matrix equation is true for $n = k$, then it is shown to be true for $n = k + 1$. As the matrix equation is true for $n = 1$, it is now also true for all $n \geqslant 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.

3 Basis: $n = 1$: LHS $= 3 \times 1 + 4 = 7$; RHS $= \frac{1}{2} \times 1(3 \times 1 + 11) = 7$

Assumption:

$$\sum_{r=1}^{k}(3r+4)=\frac{1}{2}k(3k+11)$$

Induction:

$$\sum_{r=1}^{k+1}(3r+4)=\sum_{r=1}^{k}(3r+4)+3(k+1)+4$$

$$=\frac{1}{2}k(3k+11)+3(k+1)+4=\frac{1}{2}(3k^2+17k+14)$$
$$=\frac{1}{2}(k+1)(3(k+1)+11)$$

So if the statement holds for $n = k$, it holds for $n = k + 1$.

Conclusion: The statement holds for all $n \in \mathbb{Z}^+$.

4 a $n=1$; $\text{LHS}=\begin{pmatrix}9 & 16\\ -4 & -7\end{pmatrix}^1=\begin{pmatrix}9 & 16\\ -4 & -7\end{pmatrix}$

$$=\begin{pmatrix}8(1)+1 & 16(1)\\ -4(1) & 1-8(1)\end{pmatrix}=\begin{pmatrix}9 & 16\\ -4 & -7\end{pmatrix}$$

As LHS = RHS, the matrix equation is true for n = 1.

Assume that the matrix equation is true for n = k.

i.e. $\begin{pmatrix}9 & 16\\ -4 & -7\end{pmatrix}^k=\begin{pmatrix}8k+1 & 16k\\ -4k & 1-8k\end{pmatrix}$.

With n = k + 1 the matrix equation becomes

$$\begin{pmatrix}9 & 16\\ -4 & -7\end{pmatrix}^{k+1}=\begin{pmatrix}9 & 16\\ -4 & -7\end{pmatrix}^k\begin{pmatrix}9 & 16\\ -4 & -7\end{pmatrix}$$

$$=\begin{pmatrix}8k+1 & 16k\\ -4k & 1-8k\end{pmatrix}\begin{pmatrix}9 & 16\\ -4 & -7\end{pmatrix}$$

$$=\begin{pmatrix}72k+9-64k & 128k+16-112k\\ -36k-4+32k & -64k-7+56k\end{pmatrix}$$

$$=\begin{pmatrix}8k+9 & 16k+16\\ -4k-4 & -8k-7\end{pmatrix}$$

$$=\begin{pmatrix}8(k+1)+1 & 16(k+1)\\ -4(k+1) & 1-8(k+1)\end{pmatrix}$$

Therefore the matrix equation is true when n = k + 1.

If the matrix equation is true for n = k, then it is shown to be true for n = k + 1. As the matrix equation is true for n = 1, it is now also true for all $n \geqslant 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.

b $\det(\mathbf{A}^n)=(8n+1)(1-8n)--64n^2$

$$=8n-64n^2+1-8n+64n^2$$

$$=1$$

$$\mathbf{B}=(\mathbf{A}^n)^{-1}=\frac{1}{1}\begin{pmatrix}1-8n & -16n\\ 4n & 8n+1\end{pmatrix}$$

So $\mathbf{B}=\begin{pmatrix}1-8n & -16n\\ 4n & 8n+1\end{pmatrix}$

5 **a** $f(n+1) = 5^{2(n+1)-1} + 1$

$= 5^{2n+2-1} + 1$

$= 5^{2n-1}.5^2 + 1$

$= 24(5^{2n-1}) + 1$

$\therefore f(n+1) - f(n) = [25(5^{2n-1}) + 1] - [5^{2n-1} + 1]$

$= 25(5^{2n-1}) + 1 - (5^{2n-1}) - 1$

$= 24(5^{2n-1})$

Therefore**,** μ = 24.

b $f(n) = 5^{2n-1} + 1$, where $n \in \mathbb{Z}^+$.

$\therefore f(1) = 5^{2(1)-1} + 1 = 5^1 + 1 = 6$, which is divisible by 6.

$\therefore$ f(n) is divisible by 6 when n = 1.

Assume that for n = k,

$f(k) = 5^{2k-1} + 1$ is divisible by 6 for $k \in \mathbb{Z}^+$.

Using (a) $f(k+1) - f(k) = 24(5^{2k-1})$

$\therefore f(k+1) = f(k) + 24(5^{2k-1})$

As both f(k) and $24(5^{2k-1})$ are divisible by 6 then the sum of these two terms must also be divisible by 6. Therefore f(n) is divisible by 6 when n = k + 1.

If f(n) is divisible by 6 when n = k, then it has been shown that f(n) is also divisible by 6 when n = k + 1. As f(n) is divisible by 6 when n = 1n, f(n) is also divisible by 6 for all $n \geqslant 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.

6 Let $\mathrm{f}(n) = 7^n + 4^n + 1$, where $n \in \mathbb{Z}^+$.

$\therefore \mathrm{f}(1) = 7^1 + 4^1 + 1 = 7 + 4 + 1 = 12$, which is divisible by 6.

$\therefore$ f(n) is divisible by 6 when n = 1.

Assume that for $n = k$,

$\mathrm{f}(k) = 7^k + 4^k + 1$ is divisible by 6 for $k \in \mathbb{Z}^+$.

$$\begin{aligned}\therefore \mathrm{f}(k+1) &= 7^{k+1} + 4^{k+1} + 1\\ &= 7^k.7^1 + 4^k.4^1 + 1\\ &= 7(7^k) + 4(4^k) + 1\end{aligned}$$

$$\begin{aligned}\therefore \mathrm{f}(k+1) - \mathrm{f}(k) &= [7(7^k) + 4(4^k) + 1] - [7^k + 4^k + 1]\\ &= 7(7^k) + 4(4^k) + 1 - 7^k - 4^k - 1\\ &= 6(7^k) + 3(4^k)\\ &= 6(7^k) + 3(4^{k-1}).4^1\\ &= 6(7^k) + 12(4^{k-1})\\ &= 6[7^k + 2(4)^{k-1}]\end{aligned}$$

$$\therefore \mathrm{f}(k+1) = \mathrm{f}(k) + 6[7^k + 2(4)^{k-1}]$$

As both f(k) and $6[7^k + 2(4)^{k-1}]$ are divisible by 6 then the sum of these two terms must also be divisible by 6.

Therefore f(n) is divisible by 6 when $n = k + 1$.

If f(n) is divisible by 6 when $n = k$, then it has been shown that f(n) is also divisible by 6 when $n = k + 1$. As f(n) is divisible by 6 when n = 1, f(n) is also divisible by 6 for all $n \geqslant 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.

7 <u>Basis:</u> n = 1: LHS = 1 × 5 = 5; RHS = $\frac{1}{6} \times 1 \times 2 \times (2 + 13) = 5$

<u>Assumption:</u>

$$\sum_{r=1}^{k} r(r+4) = \frac{1}{6}k(k+1)(2k+13)$$

<u>Induction:</u>

$$\begin{aligned}\sum_{r=1}^{k+1} r(r+4) &= \sum_{r=1}^{k} r(r+4) + (k+1)(k+5)\\ &= \frac{1}{6}k(k+1)(2k+13) + (k+1)(k+5)\\ &= \frac{1}{6}(2k^3 + 21k^2 + 49k + 30) = \frac{1}{6}(k+1)(k+2)(2(k+1)+13)\end{aligned}$$

So if the statement holds for $n = k$, it holds for $n = k + 1$.

<u>Conclusion:</u> The statement holds for all $n \in \mathbb{Z}^+$.

8 a Basis: $n = 1$: LHS = 1 + 4 = 5; RHS = $\frac{1}{3} \times 1 \times 3 \times 5 = 5$

Assumption:

$$\sum_{r=1}^{2k} r^2 = \frac{1}{3}k(2k+1)(4k+1)$$

Induction:

$$\sum_{r=1}^{2(k+1)} r^2 = \sum_{r=1}^{2k} r^2 + (2k+1)^2 + (2k+2)^2$$

$$= \frac{1}{3}k(2k+1)(4k+1) + (2k+1)^2 + (2k+2)^2$$

$$= \frac{1}{3}(8k^3 + 30k^2 + 37k + 15)$$

$$= \frac{1}{3}(k+1)(2(k+1)+1)(4(k+1)+1)$$

So if the statement holds for $n = k$, it holds for $n = k + 1$.

Conclusion: The statement holds for all $n \in \mathbb{Z}^+$.

b Using **a** and the formula for $\sum_{r=1}^{n} r^2$

$$\frac{1}{6} \times 2n(2n+1)(4n+1) = \frac{1}{6}kn(n+1)(2n+1)$$

$$2n(2n + 1) + (4n + 1) = kn(n + 1)(2n + 1)$$

$$16n^3 + 12n^3 + 12n^2 + 2n = k(2n^3 + 3n^2)$$

$$\Rightarrow k = \frac{2n(8n^2 + 6n + 1)}{n(2n^2 + 3n + 1)} = \frac{2(2n+1)(4n+1)}{(2n+1)(n+1)} = \frac{8n+2}{n+1}$$

$$\Rightarrow kn + k = 8n + 2 \Rightarrow n(k-8) = 2 - k \Rightarrow n = \frac{2-k}{k-8}$$

9 a Basis: $n = 1$: LHS = RHS = $\begin{pmatrix} 2c & 1 \\ 0 & c \end{pmatrix}$

Assumption:

$$\mathbf{M}^k = \mathbf{c}^k \begin{pmatrix} 2^k & \frac{2^k - 1}{c} \\ 0 & 1 \end{pmatrix}$$

Induction:

$$\mathbf{M}^{k+1} = \mathbf{M}^k \begin{pmatrix} 2c & 1 \\ 0 & c \end{pmatrix}$$

$$= c^k \begin{pmatrix} 2^k & \frac{2^k - 1}{c} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2c & 1 \\ 0 & c \end{pmatrix} = c^{k+1} \begin{pmatrix} 2^k & \frac{2^k - 1}{c} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & \frac{1}{c} \\ 0 & 1 \end{pmatrix}$$

$$= c^{k+1} \begin{pmatrix} 2^{k+1} & \frac{2^k + 2^k - 1}{c} \\ 0 & 1 \end{pmatrix} = c^{k+1} \begin{pmatrix} 2^{k+1} & \frac{2^{k+1} - 1}{c} \\ 0 & 1 \end{pmatrix}$$

So if the statement holds for $n = k$, it holds for $n = k + 1$.

Conclusion: The statement holds for all $n \in \mathbb{Z}^+$.

b Consider $n = 1$: det $\mathbf{M} = 50 \Rightarrow 2c^2 = 50$
So $c = 5$, since c is +ve.

Challenge

a Basis: $n = 1$: LHS = RHS = $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

Assumption:

$$\mathbf{M}^k = \begin{pmatrix} \cos k\theta & -\sin k\theta \\ \sin k\theta & \cos k\theta \end{pmatrix}$$

Induction:

$$\mathbf{M}^{k+1} = \mathbf{M}^k \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} \cos k\theta & -\sin k\theta \\ \sin k\theta & \cos k\theta \end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} \cos k\theta\ \cos\theta\ -\sin k\theta \sin\theta & -\cos\theta \sin\theta\ \ -\sin\theta \cos\theta \\ \sin k\theta\ \cos\theta\ +\cos k\theta \sin\theta & -\sin k\theta \sin\theta\ +\cos k\theta \cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} \cos((k+1)\theta) & -\sin((k+1)\theta) \\ \sin((k+1)\theta) & \cos((k+1)\theta) \end{pmatrix}$$

So if the statement holds for $n = k$, it holds for $n = k + 1$.

Conclusion: The statement holds for all $n \in \mathbb{Z}^+$.

b The matrix **M** represents a rotation through angle θ, and so $\mathbf{M}^n$ represents a rotation through angle $n\theta$.