Small Angle Approximations

When x is small and measured in radians

$$\sin x \approx x \,, \quad \cos x \approx 1 - \frac{x^2}{2} \,, \quad \tan x \approx x$$

Valid upto about 0.4 radians or 23°

1 When θ is small enough for θ^3 to be ignored, find approximate expressions for the following.

(i) $\dfrac{\theta\sin\theta}{1-\cos\theta}$ **(ii)** $2\cos\left(\dfrac{\pi}{3}+\theta\right)$

(iii) $\cos\theta\cos 2\theta$ **(iv)** $\dfrac{\theta\tan\theta}{1-\cos 2\theta}$

(v) $\dfrac{\cos 4\theta-\cos 2\theta}{\sin 4\theta-\sin 2\theta}$ **(vi)** $\sin(a+\theta)\sin\theta$ (Note: a is not small.)

2 **(i)** Find an approximate expression for $\sin 2\theta+\tan 3\theta$ when θ is small enough for 3θ to be considered as small.

(ii) Hence find

$$\lim_{\theta\to 0}\frac{\sin 2\theta+\tan 3\theta}{\theta}.$$

3 **(i)** Find an approximate expression for $1-\cos\theta$ when θ is small.

(ii) Hence find

$$\lim_{\theta\to 0}\frac{1-\cos\theta}{4\theta\sin\theta}.$$

4 **(i)** Find an approximate expression for $\sin\theta\left[\sin\left(\dfrac{\pi}{6}+\theta\right)-\sin\dfrac{\pi}{6}\right]$ when θ is small.

(ii) Find an approximate expression for $1-\cos 2\theta$ when θ is small.

(iii) Hence find

$$\lim_{\theta\to 0}\frac{\sin\theta\left[\sin\left(\frac{\pi}{6}+\theta\right)-\sin\frac{\pi}{6}\right]}{1-\cos 2\theta}.$$

5 **(i)** Find an approximate expression for $1-\cos 4\theta$ when θ is small enough for 4θ to be considered as small.

(ii) Find an approximate expression for $\tan^2 2\theta$ when θ is small enough for 2θ to be considered as small.

(iii) Hence find

$$\lim_{\theta\to 0}\frac{1-\cos 4\theta}{\tan^2 2\theta}.$$

1 When θ is small enough for θ^3 to be ignored, find approximate expressions for the following.

(i) $\dfrac{\theta\sin\theta}{1-\cos\theta}$ **(ii)** $2\cos\left(\dfrac{\pi}{3}+\theta\right)$

(iii) $\cos\theta\cos 2\theta$ **(iv)** $\dfrac{\theta\tan\theta}{1-\cos 2\theta}$

(v) $\dfrac{\cos 4\theta-\cos 2\theta}{\sin 4\theta-\sin 2\theta}$ **(vi)** $\sin(a+\theta)\sin\theta$ (Note: a is not small.)

1 ii) $2\cos\left(\frac{\pi}{3}+\theta\right)$

$= 2\left[\cos\frac{\pi}{3}\cos\theta - \sin\frac{\pi}{3}\sin\theta\right]$

$\approx 2\left[\frac{1}{2}\left(1-\frac{\theta^2}{2}\right) - \frac{\sqrt{3}}{2}\theta\right]$

$\approx 1-\frac{\theta^2}{2}-\sqrt{3}\theta$

v) $\dfrac{\cos 4\theta - \cos 2\theta}{\sin 4\theta - \sin 2\theta} \approx \dfrac{1-\frac{(4\theta)^2}{2}-\left(1-\frac{(2\theta)^2}{2}\right)}{4\theta-2\theta}$

$= \dfrac{1-8\theta^2-1+2\theta^2}{2\theta}$

$= \dfrac{-6\theta^2}{2\theta} = -3\theta$

i) $\dfrac{\theta\sin\theta}{1-\cos\theta} \approx \dfrac{\theta\times\theta}{1-\left(1-\frac{\theta^2}{2}\right)} = \dfrac{\theta^2}{\frac{\theta^2}{2}} = 2$

iii) $\cos\theta\cos 2\theta \approx \left(1-\frac{\theta^2}{2}\right)\left(1-\frac{(2\theta)^2}{2}\right)$

$$= \left(1-\frac{\theta^2}{2}\right)\left(1-2\theta^2\right)$$

$$= 1-\frac{\theta^2}{2}-2\theta^2+\theta^4$$

$$= 1-\frac{5\theta^2}{2}+\cancel{\theta^4}$$

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iv) $\frac{\theta\tan\theta}{1-\cos 2\theta} \approx \frac{\theta\times\theta}{1-\left(1-\frac{(2\theta)^2}{2}\right)} = \frac{\theta^2}{2\theta^2} = \frac{1}{2}$

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vi) $\sin(\alpha+\theta)\sin\theta$

$$= \left[\sin\alpha\cos\theta+\cos\alpha\sin\theta\right]\sin\theta$$

$$\approx \left[\sin\alpha\left(1-\frac{\theta^2}{2}\right)+\theta\cos\alpha\right]\theta$$

$$= \left[\sin\alpha-\frac{\theta^2}{2}\sin\alpha+\theta\cos\alpha\right]\theta$$

$$= \theta\sin\alpha\cancel{-\frac{\theta^3}{2}\sin\alpha}+\theta^2\cos\alpha$$

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2 **(i)** Find an approximate expression for $\sin 2\theta + \tan 3\theta$ when θ is small enough for 3θ to be considered as small.

(ii) Hence find

$$\lim_{\theta \to 0} \frac{\sin 2\theta + \tan 3\theta}{\theta}.$$

$$\sin 2\theta + \tan 3\theta \approx 2\theta + 3\theta = 5\theta$$

$$\lim_{\theta \to 0} \frac{\sin 2\theta + \tan 3\theta}{\theta} = \frac{5\theta}{\theta} = 5$$

3 **(i)** Find an approximate expression for $1 - \cos\theta$ when θ is small.

(ii) Hence find

$$\lim_{\theta \to 0} \frac{1 - \cos\theta}{4\theta \sin\theta}.$$

$$1 - \cos\theta \approx 1 - \left(1 - \frac{\theta^2}{2}\right)$$

$$= \frac{\theta^2}{2}$$

$$\lim_{\theta \to 0} \frac{1 - \cos\theta}{4\theta \sin\theta} = \frac{\frac{\theta^2}{2}}{4\theta^2} = \frac{1}{8}$$

4 (i) Find an approximate expression for $\sin\theta\left[\sin\left(\dfrac{\pi}{6}+\theta\right)-\sin\dfrac{\pi}{6}\right]$ when θ is small.

(ii) Find an approximate expression for $1-\cos 2\theta$ when θ is small.

(iii) Hence find

$$\lim_{\theta\to 0}\frac{\sin\theta\left[\sin\left(\frac{\pi}{6}+\theta\right)-\sin\frac{\pi}{6}\right]}{1-\cos 2\theta}.$$

$$\sin\theta\left[\sin\left(\frac{\pi}{6}+\theta\right)-\sin\frac{\pi}{6}\right]$$

$$= \sin\theta\left[\sin\frac{\pi}{6}\cos\theta+\cos\frac{\pi}{6}\sin\theta-\sin\frac{\pi}{6}\right]$$

$$\approx \theta\left[\frac{1}{2}\left(1-\frac{\theta^2}{2}\right)+\frac{\sqrt{3}}{2}\theta-\frac{1}{2}\right]$$

$$= \theta\left[\cancel{\frac{1}{2}}-\frac{\theta^2}{4}+\frac{\sqrt{3}}{2}\theta-\cancel{\frac{1}{2}}\right]$$

$$= -\frac{\theta^3}{4}+\frac{\sqrt{3}}{2}\theta^2$$

$$\lim_{\theta\to 0}\frac{\sin\theta\left[\sin\left(\frac{\pi}{6}+\theta\right)-\sin\frac{\pi}{6}\right]}{1-\cos 2\theta} = \frac{\cancel{-\frac{\theta^3}{4}}+\frac{\sqrt{3}}{2}\theta^2}{1-\left(1-\frac{(2\theta)^2}{2}\right)}$$

$$= \frac{+\frac{\sqrt{3}}{2}\theta^2}{2\theta^2}$$

$$= +\frac{\sqrt{3}}{4}$$

5 (i) Find an approximate expression for $1 - \cos 4\theta$ when θ is small enough for 4θ to be considered as small.

(ii) Find an approximate expression for $\tan^2 2\theta$ when θ is small enough for 2θ to be considered as small.

(iii) Hence find

$$\lim_{\theta \to 0} \frac{1 - \cos 4\theta}{\tan^2 2\theta}.$$

i) $1 - \cos 4\theta \approx 1 - \left(1 - \frac{(4\theta)^2}{2}\right)$

$= 8\theta^2$

ii) $\tan^2 2\theta = (\tan 2\theta)^2 \approx (2\theta)^2 = 4\theta^2$

iii) $\lim_{\theta \to 0} \frac{1 - \cos 4\theta}{\tan^2 2\theta} = \frac{8\theta^2}{4\theta^2} = 2$

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AP Assessment

Jan 11

6 The third term of an arithmetic progression is 24. The tenth term is 3.

Find the first term and the common difference.

Find also the sum of the 21st to 50th terms inclusive. **[5]**

Jan 12

10 In an arithmetic progression, the second term is 11 and the sum of the first 40 terms is 3030. Find the first term and the common difference. **[5]**

Jun 12

2 Find the second and third terms in the sequence given by

$$u_1 = 5,$$
$$u_{n+1} = u_n + 3.$$

Find also the sum of the first 50 terms of this sequence. **[4]**

6) 3rd $a + 2d = 24$ ①

10^{th} $a + 9d = 3$ ②

② - ① $7d = -21$

$\underline{d = -3}$

$a + 2(-3) = 24$

$a - 6 = 24$

$\underline{a = 30}$

10)

$a + d = 11$ ①

$S_n = \frac{n}{2}[2a + (n-1)d]$

$S_{40} = \frac{40}{2}[2a + 39d]$

$S_{40} = 40a + 780d = 3030$ ②

From ① $a = 11 - d$

Sub in ② $40(11-d) + 780d = 3030$

$440 - 40d + 780d = 3030$

$740d = 2590$

$d = \frac{2590}{740} = 3.5$

$a = 11 - 3.5$

$\underline{a = 7.5}$

$\underline{d = 3.5}$

2) $U_1 = 5$ $U_{n+1} = U_n + 3$

$U_2 = 8$

$U_3 = 11$

AP $a = 5$, $d = 3$

$$S_n = \frac{n}{2}\left[2a + (n-1)d\right]$$

$$S_{50} = \frac{50}{2}\left[10 + 49 \times 3\right] = \underline{3925}$$