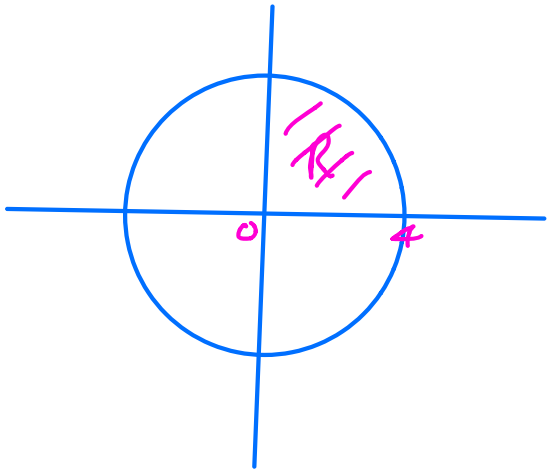
Integration Involving Parametrics



$$x^2 + y^2 = 4^2$$

Find area of R

$$\text{Area} = \int_0^4 y\,dx$$

$$x = 4\cos\theta \qquad y = 4\sin\theta$$

$$\int_0^4 y\,dx = \int_{\frac{\pi}{2}}^{0} y\frac{dx}{d\theta}\,d\theta$$

when $x = 0$
$\theta = \frac{\pi}{2}$

when $x = 4$
$\theta = 0$

$$x = 4\cos\theta$$

$$\frac{dx}{d\theta} = -4\sin\theta$$

$$\int_{\frac{\pi}{2}}^{0} y\frac{dx}{d\theta}\,d\theta = \int_{\frac{\pi}{2}}^{0} 4\sin\theta(-4\sin\theta)\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} 16\sin^2\theta\,d\theta$$

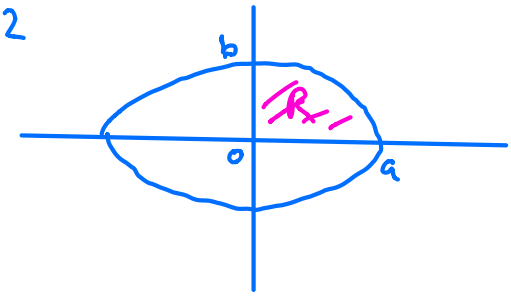
$$= 8\int_0^{\frac{\pi}{2}} (1-\cos 2\theta)\,d\theta$$

$$= 8\left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{2}}$$

$$= 8\left[\left(\frac{\pi}{2} - \frac{1}{2}\sin\pi\right) - \left(0 - \frac{1}{2}\sin 0\right)\right]$$

$$= 8 \times \frac{\pi}{2} = 4\pi$$

Ex 2



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$x = a\cos\theta \qquad y = b\sin\theta$$

$$\frac{dx}{d\theta} = -a\sin\theta$$

$$R = \int_0^a y\,dx = \int y\frac{dx}{d\theta}\,d\theta$$

$x = 0,\ \theta = \frac{\pi}{2}$

$x = a,\ \theta = 0$

$$= \int_{\frac{\pi}{2}}^{0} b\sin\theta(-a\sin\theta)\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} ab\sin^2\theta\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{ab}{2}(1-\cos 2\theta)\,d\theta$$

$$= \frac{ab}{2}\left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{2}}$$

$$= \frac{ab}{2}\left[\left(\frac{\pi}{2} - 0\right) - (0 - 0)\right]$$

$$= \frac{ab\pi}{4}$$

Area of ellipse $= 4 \times \frac{ab\pi}{4} = ab\pi$