Integration, Mixed Exercise 13

1 a $\int (x+1)(2x-5)\,dx = \int (2x^2-3x-5)\,dx$
$$= 2\frac{x^3}{3} - 3\frac{x^2}{2} - 5x + c$$
$$= \tfrac{2}{3}x^3 - \tfrac{3}{2}x^2 - 5x + c$$

b $\int \left(x^{\frac{1}{3}} + x^{-\frac{1}{3}}\right) dx = \frac{x^{\frac{4}{3}}}{\frac{4}{3}} + \frac{x^{\frac{2}{3}}}{\frac{2}{3}} + c$
$$= \tfrac{3}{4}x^{\frac{4}{3}} + \tfrac{3}{2}x^{\frac{2}{3}} + c$$

2 $f'(x) = x^2 - 3x - \frac{2}{x^2} = x^2 - 3x - 2x^{-2}$

So $f(x) = \frac{x^3}{3} - 3\frac{x^2}{2} - 2\frac{x^{-1}}{-1} + c$

So $f(x) = \frac{1}{3}x^3 - \frac{3}{2}x^2 + \frac{2}{x} + c$

But $f(1) = 1 \Rightarrow \tfrac{1}{3} - \tfrac{3}{2} + 2 + c = 1$

So $c = \tfrac{1}{6}$

So the equation is $y = \frac{1}{3}x^3 - \frac{3}{2}x^2 + \frac{2}{x} + \frac{1}{6}$

3 a $\int (8x^3 - 6x^2 + 5)\,dx = 8\frac{x^4}{4} - 6\frac{x^3}{3} + 5x + c$
$$= 2x^4 - 2x^3 + 5x + c$$

b $\int (5x+2)x^{\frac{1}{2}}\,dx = \int \left(5x^{\frac{3}{2}} + 2x^{\frac{1}{2}}\right) dx$
$$= 5\frac{x^{\frac{5}{2}}}{\frac{5}{2}} + 2\frac{x^{\frac{3}{2}}}{\frac{3}{2}} + c$$
$$= 2x^{\frac{5}{2}} + \tfrac{4}{3}x^{\frac{3}{2}} + c$$

4 $y = \frac{(x+1)(2x-3)}{\sqrt{x}}$

$y = (2x^2 - x - 3)x^{-\frac{1}{2}}$

$y = 2x^{\frac{3}{2}} - x^{\frac{1}{2}} - 3x^{-\frac{1}{2}}$

$\int y\,dx = \int \left(2x^{\frac{3}{2}} - x^{\frac{1}{2}} - 3x^{-\frac{1}{2}}\right) dx$
$$= 2\frac{x^{\frac{5}{2}}}{\frac{5}{2}} - \frac{x^{\frac{3}{2}}}{\frac{3}{2}} - 3\frac{x^{\frac{1}{2}}}{\frac{1}{2}} + c$$
$$= \tfrac{4}{5}x^{\frac{5}{2}} - \tfrac{2}{3}x^{\frac{3}{2}} - 6x^{\frac{1}{2}} + c$$

5 $\frac{dx}{dt} = (t+1)^2 = t^2 + 2t + 1$

$\Rightarrow x = \frac{t^3}{3} + 2\frac{t^2}{2} + t + c$

But $x = 0$ when $t = 2$.

So $0 = \tfrac{8}{3} + 4 + 2 + c$

$\Rightarrow c = -\tfrac{26}{3}$

So $x = \tfrac{1}{3}t^3 + t^2 + t - \tfrac{26}{3}$

When $t = 3$, $x = \tfrac{27}{3} + 9 + 3 - \tfrac{26}{3}$

So $x = \tfrac{37}{3}$ or $12\tfrac{1}{3}$

6 a $y^{\frac{1}{2}} = x^{\frac{1}{3}} + 3$

So $y = \left(x^{\frac{1}{3}} + 3\right)^2$

So $y = \left(x^{\frac{1}{3}}\right)^2 + 6x^{\frac{1}{3}} + 9$

So $y = x^{\frac{2}{3}} + 6x^{\frac{1}{3}} + 9$

$(A = 6, B = 9)$

b $\int y\,dx = \int \left(x^{\frac{2}{3}} + 6x^{\frac{1}{3}} + 9\right) dx$
$$= \frac{x^{\frac{5}{3}}}{\frac{5}{3}} + 6\frac{x^{\frac{4}{3}}}{\frac{4}{3}} + 9x + c$$
$$= \tfrac{3}{5}x^{\frac{5}{3}} + \tfrac{9}{2}x^{\frac{4}{3}} + 9x + c$$

7 a $y^{\frac{1}{2}} = 3x^{\frac{1}{4}} - 4x^{-\frac{1}{4}}$

$y = (3x^{\frac{1}{4}} - 4x^{-\frac{1}{4}})^2$

$= 9x^{\frac{1}{2}} - 24 + 16x^{-\frac{1}{2}}$

$\frac{dy}{dx} = \tfrac{9}{2}x^{-\frac{1}{2}} - 8x^{-\frac{3}{2}}$

b $\int \left(9x^{\frac{1}{2}} - 24 + 16x^{-\frac{1}{2}}\right) dx$

$= \frac{9x^{\frac{3}{2}}}{\frac{3}{2}} - 24x + \frac{16x^{\frac{1}{2}}}{\frac{1}{2}} + c$

$= 6x^{\frac{3}{2}} - 24x + 32x^{\frac{1}{2}} + c$

8 $\int \left(\frac{a}{3x^3} - ab\right) dx = \int \left(\frac{a}{3}x^{-3} - ab\right) dx$
$$= \frac{a}{3} \times \frac{x^{-2}}{-2} - abx + c$$

8 $\int\left(\frac{a}{3x^3}-ab\right)\mathrm{d}x=-\frac{a}{6x^2}-abx+c$

$= -\frac{2}{3x^2}+14x+c$

Equating coefficients $-\frac{a}{6} = -\frac{2}{3}$ and

$-ab = 14$

$a = 4$, $b = -3.5$

9 $f'(t) = -9.8t$

$f(t) = -\frac{9.8t^2}{2}+c$

$= -4.9t^2 + c$

$f(0) = -4.9(0)^2 + c$

$= 70$

$c = 70$

$f(t) = -4.9t^2 + 70$

$f(3) = -4.9(3)^2 + 70$

$= 25.9$

The height of the rock above the ground after 3 seconds is 25.9 m.

10 a $f(t) = \int(5+2t)\,\mathrm{d}t$

$=5t+\frac{2t^2}{2}+c$

$=5t+t^2+c$

As f(0) = 0, $5(0)+0^2+c=0$

$c=0$

$f(t) = 5t+t^2$

b When f(t) = 100, $5t+t^2=100$

$t^2 + 5t - 100 = 0$

Using the formula

$t = \frac{-5\pm\sqrt{5^2-4(1)(-100)}}{2(1)}$

$t = \frac{-5\pm\sqrt{425}}{2}$

$t = 7.8$ or $t = -12.8$

As $t > 0$, $t = 7.8$ seconds

11 a $2=5+2x-x^2$

$\Rightarrow x^2-2x-3=0$

$\Rightarrow (x-3)(x+1)=0$

$\Rightarrow x=-1(A),\ 3(B)$

b Area of $R=\int_{-1}^{3}(5+2x-x^2-2)\mathrm{d}x$

$=\int_{-1}^{3}(3+2x-x^2)\mathrm{d}x$

$=\left(3x+x^2-\tfrac{1}{3}x^3\right)_{-1}^{3}$

$=\left(9+9-\tfrac{27}{3}\right)-\left(-3+1+\tfrac{1}{3}\right)$

$=9+2-\tfrac{1}{3}$

$=10\tfrac{2}{3}$

12 a $\left(x^{\frac{1}{2}}-4\right)\left(x^{-\frac{1}{2}}-1\right)$

$=1-4x^{-\frac{1}{2}}-x^{\frac{1}{2}}+4=5-4x^{-\frac{1}{2}}-x^{\frac{1}{2}}$

$\int\left(x^{\frac{1}{2}}-4\right)\left(x^{-\frac{1}{2}}-1\right)\mathrm{d}x=5x-\frac{4x^{\frac{1}{2}}}{\frac{1}{2}}-\frac{x^{\frac{3}{2}}}{\frac{3}{2}}+c$

$=5x-8x^{\frac{1}{2}}-\tfrac{2}{3}x^{\frac{3}{2}}+c$

b $\int_{1}^{4}\left(x^{\frac{1}{2}}-4\right)\left(x^{-\frac{1}{2}}-1\right)\mathrm{d}x=\left(5x-8x^{\frac{1}{2}}-\tfrac{2}{3}x^{\frac{3}{2}}\right)_{1}^{4}$

$=\left(20-8\times2-\tfrac{2}{3}\times2^3\right)-\left(5-8-\tfrac{2}{3}\right)$

$=4-\tfrac{16}{3}+3+\tfrac{2}{3}$

$=7-\tfrac{14}{3}$

$=\tfrac{7}{3}$ or $2\tfrac{1}{3}$

13 a $(x-3)^2=x^2-6x+9$

So $x(x-3)^2=x^3-6x^2+9x$

$y=0\Rightarrow x=0$ or 3 (twice)

So A is the point $(3, 0)$.

b $\frac{dy}{dx}=0\Rightarrow 0=3x^2-12x+9$

$\Rightarrow 0=3(x^2-4x+3)$

$\Rightarrow 0=3(x-3)(x-1)$

$\Rightarrow 0=1$ or 3

$x=3$ at A, the minimum, so B is $(1,4)$

(Found by substituting $x = 1$ into original equation.)

13 c Area of $R = \int_0^3 \left(x^3 - 6x^2 + 9x\right)\,dx$

$= \left(\frac{1}{4}x^4 - 2x^3 + \frac{9}{2}x^2\right)_0^3$

$= \left(\frac{81}{4} - 2\times27 + \frac{9}{2}\times9\right) - (0)$

$= 6\frac{3}{4}$

14 a $y = 3x^{\frac{1}{2}} - 4x^{-\frac{1}{2}}$

$\frac{dy}{dx} = \frac{3}{2}x^{-\frac{1}{2}} + \frac{1}{2}\times4x^{-\frac{3}{2}}$

$\frac{dy}{dx} = \frac{3}{2}x^{-\frac{1}{2}} + 2x^{-\frac{3}{2}}$

b $\int y\,dx = \int\left(3x^{\frac{1}{2}} - 4x^{-\frac{1}{2}}\right)dx$

$= \frac{3x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{4x^{\frac{1}{2}}}{\frac{1}{2}} + c$

$= 2x^{\frac{3}{2}} - 8x^{\frac{1}{2}} + c$

c $\int_1^3 y\,dx = \left(2x^{\frac{3}{2}} - 8x^{\frac{1}{2}}\right)_1^3$

$= \left(2\times3\sqrt{3} - 8\sqrt{3}\right) - (2-8)$

$= -2\sqrt{3} + 6$

$= 6 - 2\sqrt{3}$

So $A = 6$ and $B = -2$

15 a $y = 12x^{\frac{1}{2}} - x^{\frac{3}{2}}$

$\frac{dy}{dx} = 6x^{-\frac{1}{2}} - \frac{3}{2}x^{\frac{1}{2}}$

$= \frac{3}{2}x^{-\frac{1}{2}}(4-x)$

b $\frac{dy}{dx} = 0 \Rightarrow x = 4,\ y = 12\times2 - 2^3 = 16$

So B is the point $(4, 16)$.

c Area $= \int_0^{12}\left(12x^{\frac{1}{2}} - x^{\frac{3}{2}}\right)dx$

$= \left(\frac{12x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{x^{\frac{5}{2}}}{\frac{5}{2}}\right)_0^{12}$

$= \left(8x^{\frac{3}{2}} - \frac{2}{5}x^{\frac{5}{2}}\right)_0^{12}$

$= \left(8\times\sqrt{12^3} - \frac{2}{5}\sqrt{12^5}\right) - (0)$

$= 133$ (3 s.f.)

16 a $x(8-x) = 12$

$\Rightarrow 8x - x^2 = 12$

$\Rightarrow 0 = x^2 - 8x + 12$

$\Rightarrow 0 = (x-6)(x-2)$

$\Rightarrow x = 2$ or $x = 6$

M is on the same line as L.

So M is the point $(6, 12)$.

b Area $= \int_6^8\left(8x - x^2\right)dx$

$= \left(4x^2 - \frac{x^3}{3}\right)_6^8$

$= \left(4\times64 - \frac{512}{3}\right) - \left(4\times36 - \frac{216}{3}\right)$

$= 256 - 170\frac{2}{3} - 144 + 72$

$= 13\frac{1}{3}$

17 a A is the point $(1, 0)$, B is the point $(5, 0)$.

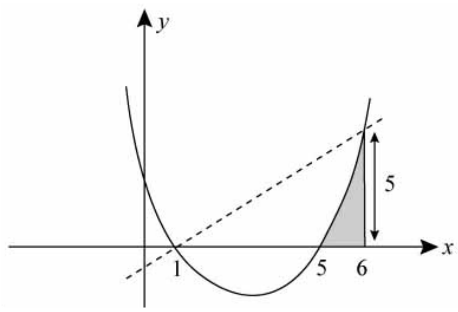
$x - 1 = (x-1)(x-5)$

$\Rightarrow 0 = (x-1)(x-5-1)$

$\Rightarrow 0 = (x-1)(x-6)$

$\Rightarrow x = 1,\ x = 6$

So C is the point $(6, 5)$.

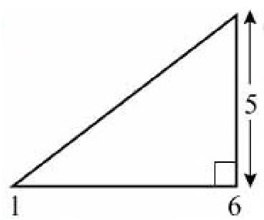


b Drop a perpendicular from C to the x-axis to a point D.

The area of the shaded region is

Area of triangle ABD $\int_5^6 (x-1)(x-5)\ dx$

= Area of ABD $-\int_5^6\left(x^2 - 6x + 5\right)dx$



17 b $\text{Area} = \left(\frac{1}{2}\times5\times5\right) - \int_5^6\left(x^2-6x+5\right)\,dx$

$= 12\frac{1}{2} - \left[\frac{1}{3}x^3 - 3x^2 + 5x\right]_5^6$

$= 12\frac{1}{2} - \left[\left(72-108+30\right) - \left(41\frac{2}{3} - 75 + 25\right)\right]$

$= 12\frac{1}{2} - (-6) - \left(-8\frac{1}{3}\right)$

$= 12\frac{1}{2} + 6 + 8\frac{1}{3}$

$= 26\frac{5}{6}$

18 a For the point A, which lies on the line and the curve

$4q + 25 = p + 40 - 16$

$\Rightarrow\ 4q = p - 1 \quad \textbf{(1)}$

For the point B, which lies on the line and the curve

$8q + 25 = p + 80 - 64$

$\Rightarrow\ 8q = p - 9 \quad \textbf{(2)}$

Subtracting $\textbf{(2)} - \textbf{(1)}$

$\Rightarrow 4q = -8$

$\Rightarrow q = -2$

Substituting into **(1)**

$\Rightarrow p = 1 + 4q$

$\Rightarrow p = -7$

b At A, $y = 4q + 25 = 17$

So C is given by

$17 = -7 + 10x - x^2$

$x^2 - 10x + 24 = 0$

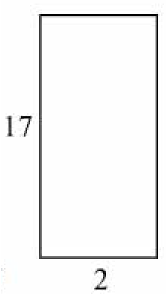
$(x-6)(x-4) = 0$

$x = 4,\ x = 6$

So C is the point $(6, 17)$

c The required area is

$\int_4^6 (-7 + 10x - x^2)\,dx - \text{area of rectangle}$



18 c $\text{Area} = \left(-7x + 5x^2 - \frac{1}{3}x^3\right)_4^6 - 34$

$= \left(-42 + 180 - 72\right) - \left(-28 + 80 - \frac{64}{3}\right) - 34$

$= \frac{4}{3}$ or $1\frac{1}{3}$

19 $\int\left(\frac{9}{x^2} - 8\sqrt{x} + 4x - 5\right)dx$

$= \int (9x^{-2} - 8x^{\frac{1}{2}} + 4x - 5)\,dx$

$= \frac{9x^{-1}}{-1} - \frac{8x^{\frac{3}{2}}}{\frac{3}{2}} + \frac{4x^2}{2} - 5x + c$

$= -\frac{9}{x} - \frac{16x^{\frac{3}{2}}}{3} + 2x^2 - 5x + c$

20 $A^2 = \int_4^9\left(\frac{3}{\sqrt{x}} - A\right)dx$

$= \int_4^9\left(3x^{-\frac{1}{2}} - A\right)dx$

$= \left[\frac{3x^{\frac{1}{2}}}{\frac{1}{2}} - Ax\right]_4^9$

$= \left[6x^{\frac{1}{2}} - Ax\right]_4^9$

$= \left(6(9)^{\frac{1}{2}} - A(9)\right) - \left(6(4)^{\frac{1}{2}} - A(4)\right)$

$= (18 - 9A) - (12 - 4A)$

$0 = (A + 6)(A - 1)$

$A = -6$ or $A = 1$

21 a $f'(x) = \frac{(2-x^2)^3}{x^2}$

$= \frac{(2-x^2)(2-x^2)(2-x^2)}{x^2}$

$= \frac{(4-4x^2+x^4)(2-x^2)}{x^2}$

$= x^{-2}(8 - 12x^2 + 6x^4 - x^6)$

$= 8x^{-2} - 12 + 6x^2 - x^4$

So $A = 6$ and $B = -1$

b $f''(x) = -16x^{-3} + 12x - 4x^3$

c $f(x) = \int (8x^{-2} - 12 + 6x^2 - x^4)\,dx$

$= \frac{8x^{-1}}{-1} - 12x + \frac{6x^3}{3} - \frac{x^5}{5} + c$

$= -\frac{8}{x} - 12x + 2x^3 - \frac{x^5}{5} + c$

21 c When $x = -2$ and $y = 9$

$$-\frac{8}{-2} - 12(-2) + 2(-2)^3 - \frac{(-2)^5}{5} + c = 9$$
$$4 + 24 - 16 + \tfrac{32}{5} + c = 9$$
$$c = -\tfrac{47}{5}$$
$$f(x) = -\frac{8}{x} - 12x + 2x^3 - \frac{x^5}{5} - \frac{47}{5}$$

22 a $y = 3 - 5x - 2x^2$

When $y = 0$, $3 - 5x - 2x^2 = 0$

$(3 + x)(1 - 2x) = 0$

$x = -3$ or $x = \tfrac{1}{2}$

The points are $A(-3, 0)$ and $B(\tfrac{1}{2}, 0)$.

b $\int_{-3}^{\frac{1}{2}} (3 - 5x - 2x^2)\,dx$

$$= \left[3x - \frac{5x^2}{2} - \frac{2x^3}{3}\right]_{-3}^{\frac{1}{2}}$$
$$= \left(3\left(\tfrac{1}{2}\right) - \frac{5\left(\frac{1}{2}\right)^2}{2} - \frac{2\left(\frac{1}{2}\right)^3}{3}\right) - \left(3(-3) - \frac{5(-3)^2}{2} - \frac{2(-3)^3}{3}\right)$$
$$= \left(\tfrac{3}{2} - \tfrac{5}{8} - \tfrac{1}{12}\right) - \left(-9 - \tfrac{45}{2} + \tfrac{54}{3}\right)$$
$$= 14\tfrac{7}{24}$$

23 a $(x - 4)(2x + 3) = 0$

$x = 4$ or $x = -\tfrac{3}{2}$

The points are $A(-\tfrac{3}{2}, 0)$ and $B(4, 0)$.

b $R = \int_{-\frac{3}{2}}^{4} (x - 4)(2x + 3)\,dx$

$$= \int_{-\frac{3}{2}}^{4} (2x^2 - 5x - 12)\,dx$$
$$= \left[\frac{2x^3}{3} - \frac{5x^2}{2} - 12x\right]_{-\frac{3}{2}}^{4}$$
$$= \left(\frac{2(4)^3}{3} - \frac{5(4)^2}{2} - 12(4)\right) - \left(\frac{2(-\frac{3}{2})^3}{3} - \frac{5(-\frac{3}{2})^2}{2} - 12(-\tfrac{3}{2})\right)$$
$$= \left(\tfrac{128}{3} - 40 - 48\right) - \left(-\tfrac{9}{4} - \tfrac{45}{8} + 18\right)$$
$$= -55\tfrac{11}{24}$$

Area = $55\tfrac{11}{24}$

24 a $x(x - 3)(x + 2) = 0$

$x = 0$, $x = 3$ or $x = -2$

The points are $A(-2, 0)$ and $B(3, 0)$.

b $\int_{-2}^{0} x(x-3)(x+2)\,dx - \int_{0}^{3} x(x-3)(x+2)\,dx$

$$= \int_{-2}^{0} (x^3 - x^2 - 6x)\,dx - \int_{0}^{3} (x^3 - x^2 - 6x)\,dx$$

$$\int_{-2}^{0} (x^3 - x^2 - 6x)\,dx = \left[\frac{x^4}{4} - \frac{x^3}{3} - \frac{6x^2}{2}\right]_{-2}^{0}$$
$$= \left[\frac{x^4}{4} - \frac{x^3}{3} - 3x^2\right]_{-2}^{0}$$
$$= \left(\frac{0^4}{4} - \frac{0^3}{3} - 3(0)^2\right) - \left(\frac{(-2)^4}{4} - \frac{(-2)^3}{3} - 3(-2)^2\right)$$
$$= 0 - \left(4 + \tfrac{8}{3} - 12\right)$$
$$= 5\tfrac{1}{3}$$

$$\int_{0}^{3} (x^3 - x^2 - 6x)\,dx = \left[\frac{x^4}{4} - \frac{x^3}{3} - 3x^2\right]_{0}^{3}$$
$$= \left(\frac{3^4}{4} - \frac{3^3}{3} - 3(3)^2\right) - \left(\frac{0^4}{4} - \frac{0^3}{3} - 3(0)^2\right)$$
$$= \left(\tfrac{81}{4} - 9 - 27\right)$$
$$= -15\tfrac{3}{4}$$

Total area is $5\tfrac{1}{3} - (-15\tfrac{3}{4}) = 21\tfrac{1}{12}$

Challenge

To find the points of intersection:

$x^2 - 5x + 7 = \frac{1}{2}x^2 - \frac{5}{2}x + 7$
$2x^2 - 10x + 14 = x^2 - 5x + 14$
$x^2 - 5x = 0$
$x(x - 5) = 0$
$x = 0$ or $x = 5$

Area R =
(area under the curve $y = \frac{1}{2}x^2 - \frac{5}{2}x + 7$)
– (area under the curve $y = x^2 - 5x + 7$)

Area under the curve: $y = \frac{1}{2}x^2 - \frac{5}{2}x + 7$:

$$\text{Area under the curve} = \int_0^5 \left(\tfrac{1}{2}x^2 - \tfrac{5}{2}x + 7\right) \mathrm{d}x$$
$$= \left[\frac{1}{3}\times\frac{x^3}{2} - \frac{1}{2}\times\frac{5x^2}{2} + 7x\right]_0^5$$
$$= \left[\frac{x^3}{6} - \frac{5x^2}{4} + 7x\right]_0^5$$
$$= \left(\frac{5^3}{6} - \frac{5(5)^2}{4} + 7(5)\right) - \left(\frac{0^3}{6} - \frac{5(0)^2}{4} + 7(0)\right)$$
$$= \left(\tfrac{125}{6} - \tfrac{125}{4} + 35\right)$$
$$24\tfrac{7}{12}$$

Area under the curve: $y = x^2 - 5x + 7$:

$$\text{Area under the curve} = \int_0^5 (x^2 - 5x + 7)\, \mathrm{d}x$$
$$= \left[\frac{x^3}{3} - \frac{5x^2}{2} + 7x\right]_0^5$$
$$= \left(\frac{5^3}{3} - \frac{5(5)^2}{2} + 7(5)\right) - \left(\frac{0^3}{3} - \frac{5(0)^2}{2} + 7(0)\right)$$
$$= \left(\tfrac{125}{3} - \tfrac{125}{2} + 35\right)$$
$$= 14\tfrac{1}{6}$$

$$R = 24\tfrac{7}{12} - 14\tfrac{1}{6}$$
$$= 10\tfrac{5}{12}$$