Poisson distributions Mixed exercise 2

1 a Let X be the number of accidents in a one-month period. Assume a Poisson distribution.
So $X \sim \text{Po}(0.7)$

$\text{P}(X=0)=\text{e}^{-0.7}=0.4966$ (4 d.p.)

b Let Y be the number of accidents in a three-month period. So $Y \sim \text{Po}(2.1)$

$$\text{P}(Y=2)=\frac{\text{e}^{-2.1}\times 2.1^2}{2!}=0.2700 \text{ (4 d.p.)}$$

c Let A be the number of accident-free months in a sample of six months. So $A \sim \text{B}(6, 0.4966)$ from part **a**

$$\text{P}(A=2)=\binom{6}{2}\times(0.4966)^2\times(0.5034)^4=0.2376 \text{ (4 d.p.)}$$

2 a X must have a constant mean rate so that the mean number of misprints in a sample is proportional to the number of chapters. Misprints must occur independently of one another and be distinct (i.e. can be counted singly in the text).

b The model is $X \sim \text{Po}(2.25)$. Find $\text{P}(X \leqslant 1)$ using a calculator:

$\text{P}(X \leqslant 1)=0.3425$

c Let Y be the number of misprints in two randomly chosen chapters. So $Y \sim \text{Po}(4.5)$
As $\lambda=4.5$, the required value can be found from the tables in the textbook:

$$\begin{aligned}\text{P}(Y>6)&=1-\text{P}(Y\leqslant 6)\\&=1-0.8311=0.1689\end{aligned}$$

3 As $Y \sim \text{Po}(\lambda)$ then $\text{P}(Y=5)=\dfrac{\text{e}^{-\lambda}\lambda^5}{5!}$ and $\text{P}(Y=3)=\dfrac{\text{e}^{-\lambda}\lambda^3}{3!}$

So as $\text{P}(Y=5)=1.25\times\text{P}(Y=3)$

$$\frac{\text{e}^{-\lambda}\lambda^5}{120}=1.25\times\frac{\text{e}^{-\lambda}\lambda^3}{6}$$

$$\lambda^2=25$$

$$\lambda=5 \qquad \text{(since } \lambda \text{ must be positive)}$$

4 a The event (receipt of an email) has a constant mean rate through time. Emails are received singly and independently of each other.

b i Let X be the number of emails received in a 10-minute period. Assume a Poisson distribution, so $X \sim \text{Po}(6)$

$$\text{P}(X=7)=\frac{\text{e}^{-6}\times 6^7}{7!}=0.1377 \text{ (4 d.p.)}$$

ii Using the tables:

$\text{P}(X \geqslant 8)=1-\text{P}(X\leqslant 7)=1-0.7440=0.2560$

5 a A binomial distribution $\text{B}(n, p)$ may be approximated by a Poisson distribution $\text{Po}(np)$ when n is large and p is small, and typically $np \leqslant 10$.

b $X \sim \text{B}(50, 0.08)$

$$\begin{aligned}\text{P}(X\leqslant 3)&=\binom{50}{3}(0.08)^3(0.92)^{47}+\binom{50}{2}(0.08)^2(0.92)^{48}+\binom{50}{1}(0.08)(0.92)^{49}+\binom{50}{0}(0.92)^{50}\\&=0.4253 \text{ (4 d.p.)}\end{aligned}$$

5 c Use as an approximation $X \sim \text{Po}(50 \times 0.08)$, i.e. $X \sim \text{Po}(4)$
Using the tables:
$\text{P}(X \leqslant 3) = 0.4335$

d Percentage error $= \dfrac{0.4335 - 0.4253}{0.4253} \times 100\% = 1.93\%$ (2 d.p)

6 $\text{P}(Y > 10) < 0.1$ so $\text{P}(Y \leqslant 10) > 0.9$
From the tables, for $\lambda = 7$, $\text{P}(Y \leqslant 10) = 0.9015$
For $\lambda = 8$, $\text{P}(Y \leqslant 10) = 0.8159$
So the largest integer value for λ satisfying the given condition is $\lambda = 7$

7 a Let X be the number of cuttings taking root in a sample of 20. So $X \sim \text{B}(20, 0.075)$

i $\text{P}(X = 2) = \binom{20}{2}(0.075)^2(0.925)^{18} = 0.2627$ (4 d.p.)

ii $\text{P}(X > 4) = 1 - \text{P}(X \leqslant 4)$
$= 1 - 0.9858 = 0.0142$

b Let Y be the number of cuttings taking root in a sample of 80. Then $Y \sim \text{B}(80, 0.075)$ and this can be approximated by $Y \sim \text{Po}(80 \times 0.075)$, i.e. $Y \sim \text{Po}(6)$
Using the tables:
$\text{P}(Y \geqslant 8) = 1 - \text{P}(Y \leqslant 7)$
$= 1 - 0.7440 = 0.2560$

8 Let X be the number of fish caught in a two-hour period. So $X \sim \text{Po}(4)$. Use this to find the probability that the angler catches at least 5 fish on a two-hour fishing trip.
$\text{P}(X \geqslant 5) = 1 - \text{P}(X \leqslant 4) = 1 - 0.6288 = 0.3712$

Then let Y be the number of trips in which the angle catches at least 5 fish from a sample of 5 trips. So using the probability for catching at least 5 fish on a single trip, $Y \sim \text{B}(5, 0.3712)$
$\text{P}(Y = 3) = \binom{5}{3}(0.3712)^3(0.6288)^2 = 0.2022$ (4 d.p.)

9 a Let X be the number of cherries in a cake. So $X \sim \text{Po}(2.5)$

i $\text{P}(X = 4) = \dfrac{\text{e}^{-2.5} \times 2.5^4}{4!} = 0.1336$ (4 d.p.)

ii Using the tables:
$\text{P}(X \geqslant 3) = 1 - \text{P}(X \leqslant 2)$
$= 1 - 0.5438 = 0.4562$

b Let Y be the number of cherries in 4 cakes. So $Y \sim \text{Po}(10)$
Using the tables:
$\text{P}(Y > 12) = 1 - \text{P}(X \leqslant 12)$
$= 1 - 0.7916 = 0.2084$

c Let A be the number of packets containing more than 12 cherries in a sample of 8 packets. So, using the result from **b**, $A \sim \text{B}(8, 0.2084)$
$\text{P}(A = 2) = \binom{8}{2}(0.2084)^2(0.7916)^6 = 0.2992$ (4 d.p.)

10 a Let X be the number of cars sold in a week. A plausible model for number of cars sold in a week would be $X \sim \text{Po}(6)$.
It may be supposed that sales are independent of each other and occur singly (assuming the salesman does not supply businesses); the constant mean rate is consistent with a Poisson model.

b $\text{P}(X=5)=\dfrac{e^{-6}\times 6^5}{5!}=0.1606$ (4 d.p.)

c Let Y be the number of weeks in which the salesman sells exactly 5 cars, in a sample of 4 consecutive weeks. So, using the result from **b**, $Y \sim \text{B}(4, 0.1606)$

$$\text{P}(Y=2)=\binom{4}{2}(0.1606)^2(0.8394)^2=0.1090 \text{ (4 d.p.)}$$

11 Assuming a Poisson distribution for each, A with and B being the number of letters received by Abbie and Ben in a given day (respectively): $A \sim \text{Po}(1.2)$ and $B \sim \text{Po}(0.8)$ so $A+B \sim \text{Po}(2)$

a
$$\begin{aligned}\text{P}(A \geqslant 1 \text{ and } B \geqslant 1) &= \text{P}(A \geqslant 1)\times \text{P}(B \geqslant 1) \qquad \text{(independence)}\\ &=(1-\text{P}(A=0))\times(1-\text{P}(B=0))\\ &=(1-e^{-1.2})\times(1-e^{-0.8})\\ &=(1-0.30119)\times(1-0.44933)\\ &=0.69881\times 0.55067=0.3848 \text{ (4 d.p.)}\end{aligned}$$

b $\text{P}(A+B=3)=\dfrac{e^{-2}\times 2^3}{3!}=0.1804$ (4 d.p.)

c Let Y be the number of days on which they receive a total of 3 letters between them, from a sample of 5 days. So, using the result from **b**, $Y \sim \text{B}(5, 0.1804)$

$$\begin{aligned}\text{P}(Y \geqslant 3)&=1-\text{P}(Y \leqslant 2)\\ &=1-0.9560=0.0440\end{aligned}$$

12 Let X be the number of desktops sold in a day and Y the number of laptops sold in a day. Assuming Poisson distributions for both and that sales of desktops and laptops are independent: $X \sim \text{Po}(2.4)$ and $Y \sim \text{Po}(1.6)$ so $X+Y \sim \text{Po}(4)$

a
$$\begin{aligned}\text{P}(X \geqslant 2 \text{ and } Y \geqslant 2) &= \text{P}(X \geqslant 2)\times \text{P}(Y \geqslant 2) \qquad \text{(independence)}\\ &=(1-\text{P}(X \leqslant 1))\times(1-\text{P}(Y \leqslant 1))\\ &=(1-0.30844)\times(1-0.52493)\\ &=0.69156\times 0.47507=0.3285 \text{ (4 d.p.)}\end{aligned}$$

b $\text{P}(X+Y=6)=\dfrac{e^{-4}\times 4^6}{6!}=0.1042$ (4 d.p.)

c Let A be the combined total of computer sales in a two-day period. So $A \sim \text{Po}(8)$ and the required value can be found from the tables:
$\text{P}(A \leqslant 6)=0.3134$

13 a Let X be the number of booked passengers not turning up for the flight. Assuming independence between booked passengers, $X \sim \text{B}(150, 0.04)$
(Note: independence of events is doubtful in this example; consider reasons why this may be the case.)

b Approximate the binomial by $X \sim \text{Po}(150\times 0.04)$, i.e. $X \sim \text{Po}(6)$ and use tables to find:
$\text{P}(X \leqslant 1)=0.0174$

13 c $P(X \geqslant 3) = 1 - P(X \leqslant 2)$
$= 1 - 0.0620 = 0.9380$

14 a Let X be the number of misdirected calls in a sample of 10 consecutive calls. Assuming independence between calls, $X \sim B(10, 0.02)$

$$P(X > 1) = 1 - P(X \leqslant 1)$$
$$= 1 - \binom{10}{1}(0.02)^1(0.98)^9 - \binom{10}{0}(0.98)^{10}$$
$$= 0.0162 \text{ (4 d.p.)}$$

b Let Y be the number of misdirected calls in a sample of 500 calls. Again assuming independence between calls, $X \sim B(10, 0.02)$

$E(Y) = 500 \times 0.02 = 10$ and $\text{Var}(Y) = 500 \times 0.02 \times 0.98 = 9.8$

c Approximating the binomial with a Poisson distribution, $Y \sim \text{Po}(10)$, and finding the required value from tables:

$P(Y \leqslant 7) = 0.2202$

15 a Let X be the number of people with the disease in a random sample of 10 people. So $X \sim B(10, 0.025)$

$$P(X = 2) = \binom{10}{2}(0.025)^2(0.975)^8 = 0.0230 \text{ (4 d.p.)}$$

b Let Y be the number of people with the disease in a random sample of 120 people. So $Y \sim B(120, 0.025)$

$E(Y) = 120 \times 0.025 = 3$ and $\text{Var}(Y) = 120 \times 0.025 \times 0.975 = 2.925$

c Approximating the binomial with a Poisson distribution, $Y \sim \text{Po}(3)$, and finding the required value from tables:

$$P(Y > 6) = 1 - P(Y \leqslant 6)$$
$$= 1 - 0.9665 = 0.0335$$

16 a Assuming that accidents can be modelled using a Poisson distribution (so that the mean number of accidents in a given period of time will be proportional to the length of time), let X be the number of accidents in a six-month period. So $X \sim \text{Po}(7.5)$, and from the tables:

$P(X \leqslant 4) = 0.1321$

b Let Y be the number of accidents in a single month. So $Y \sim \text{Po}(1.25)$

$$P(Y \geqslant 1) = 1 - P(Y = 0)$$
$$= 1 - e^{-1.25} = 1 - 0.2865 = 0.7135 \text{ (4 d.p.)}$$

c Let A be the number of months in which there is at least one accident, out of a sample of 6 months. From part b, $A \sim B(6, 0.7135)$

$$P(A = 4) = \binom{6}{4}(0.7135)^4(0.2865)^2 = 0.3191 \text{ (4 d.p.)}$$

17 a Assume a Poisson distribution for breakdowns. Let X be the number of breakdowns in a single month, so $X \sim \text{Po}\left(\frac{2}{3}\right)$

$$\text{P}(X=2)=\frac{e^{-\frac{2}{3}}\times\left(\frac{2}{3}\right)^2}{2!}=0.1141 \text{ (4 d.p.)}$$

b Let Y be the number of months in which there are at least 2 breakdowns.

$\text{P}(X \geqslant 2)=1-\text{P}(X \leqslant 1)=1-0.8557=0.1443$

So $Y \sim \text{B}(4, 0.1443)$

$$\text{P}(Y=3)=\binom{4}{3}(0.1443)^3(0.8557)^1=0.0103 \text{ (4 d.p.)}$$

18 a Visits can be counted singly; assuming visits are independent and at a constant average rate, they may be modelled by a Poisson distribution; the rate of 240 per hour would then scale to a mean rate of 4 in any given minute. Let X be the number of visits in a single minute. So $X \sim \text{Po}(4)$

b $\text{P}(X=8)=\dfrac{e^{-4}\times 4^8}{8!}=0.0298 \text{ (4 d.p.)}$

c Let Y be the number of visits in a two-minute period. So $Y \sim \text{Po}(8)$

Using the tables:

$$\begin{aligned}\text{P}(Y \geqslant 10)&=1-\text{P}(Y \leqslant 9)\\&=1-0.7166=0.2843\end{aligned}$$

19 a Let X be the number of policies sold in the week.

$$\bar{x}=\frac{\sum fx}{\sum f}=\frac{10\times0+23\times1+35\times2+33\times3+24\times4+14\times5+7\times6+3\times7+1\times8}{10+23+35+33+24+14+7+3+1}=2.86$$

$$\sigma^2=\frac{\sum fx^2}{\sum f}-\left(\bar{x}\right)^2$$

$$=\frac{10\times0^2+23\times1^2+35\times2^2+33\times3^2+24\times4^2+14\times5^2+7\times6^2+3\times7^2+1\times8^2}{10+23+35+33+24+14+7+3+1}-2.86^2$$

$$=2.867 \text{ (3 d.p.)}$$

b The values of mean and variance are the same, to 2 significant figures, which would support the validity of a Poisson model.

c Model $X \sim \text{Po}(2.9)$ and by calculator:

$\text{P}(X \leqslant 2)=0.4460 \text{ (4 d.p.)}$

Challenge

a Assuming the number of planes landing in a given period of time can be modelled by a Poisson distribution, let X be the number of planes landing between 2 pm and 2:30 pm and let Y be the number of planes landing between 2.30 pm and 3 pm. Then $X \sim \text{Po}(7.5)$, $Y \sim \text{Po}(7.5)$ and $X+Y \sim \text{Po}(15)$

The solution uses the formula for conditional probability: $\text{P}(A \mid B) = \dfrac{\text{P}(A \text{ and } B)}{\text{P(B)}}$

$$\text{P}(X=5 \mid X+Y=10) = \frac{\text{P}(X=5 \text{ and } Y=5)}{\text{P}(X+Y=10)}$$

$$= \frac{\dfrac{e^{-7.5}\times 7.5^5}{5!}\times\dfrac{e^{-7.5}\times 7.5^5}{5!}}{\dfrac{e^{-15}\times 15^{10}}{10!}}$$

$$= \frac{10!}{(5!)^2}\times\frac{7.5^{10}}{15^{10}} = \frac{10\times 9\times 8\times 7\times 6}{5\times 4\times 3\times 2}\times\left(\frac{1}{2}\right)^{10} = \frac{3\times 2\times 7\times 6}{2^{10}} = \frac{252}{2^{10}} = \frac{63}{2^8}$$

$$= \frac{63}{256} = 0.2461 \text{ (4 d.p.)}$$

b $$\text{P}(X>7 \mid X+Y=10) = \frac{\text{P}(X>7 \text{ and } X+Y=10)}{\text{P}(X+Y=10)}$$

$$= \frac{\text{P}(X=8 \text{ and } Y=2)+\text{P}(X=9 \text{ and } Y=1)+\text{P}(X=10 \text{ and } Y=0)}{\text{P}(X+Y=10)}$$

$$= \frac{\dfrac{e^{-7.5}\times 7.5^8}{8!}\times\dfrac{e^{-7.5}\times 7.5^2}{2!}}{\dfrac{e^{-15}\times 15^{10}}{10!}} + \frac{\dfrac{e^{-7.5}\times 7.5^9}{9!}\times\dfrac{e^{-7.5}\times 7.5^1}{9!}}{\dfrac{e^{-15}\times 15^{10}}{10!}} + \frac{\dfrac{e^{-7.5}\times 7.5^{10}}{10!}\times\dfrac{e^{-7.5}\times 7.5^0}{0!}}{\dfrac{e^{-15}\times 15^{10}}{10!}}$$

$$= \frac{7.5^{10}}{15^{10}}\times 10!\times\left(\frac{1}{2\times 8!}+\frac{1}{9!}+\frac{1}{10!}\right) = \frac{1}{2^{10}}\times(45+10+1)$$

$$= \frac{7}{128} = 0.0547 \text{ (4 d.p.)}$$

An alternative approach to this problem is to treat the 10 landings as each independently having a 0.5 chance of being in the first or second half of the hour and modelling them as a binomial. Let A be the number of landings in the first half hour (2 pm to 2:30 pm) and $A \sim \text{B}(10, 0.5)$. To answer part **a**, find $\text{P}(A=5)$; to answer part **b**, find $\text{P}(A>7)$.

$$\text{P}(A=5) = \binom{10}{5}(0.5)^5(0.5)^5 = \frac{10!}{5!\times 5!}\times\frac{1}{2^{10}} = \frac{63}{256}$$

$$\text{P}(A>7) = \binom{10}{8}(0.5)^8(0.5)^2 + \binom{10}{9}(0.5)^9(0.5)^1 + \binom{10}{10}(0.5)^{10} = \frac{1}{2^{10}}\times 10!\times\left(\frac{1}{2\times 8!}+\frac{1}{9!}+\frac{1}{10!}\right) = \frac{7}{128}$$