Rates Of Change

This topic makes use of the chain rule

The radius r cm of a circular ripple made by dropping a stone into a pond is increasing at a rate of 8 cm s^{-1}. At what rate is the Area A cm^2 enclosed by the ripple increasing when the radius is 25 cm

$$A = \pi r^2$$

$$\frac{dA}{dr} = 2\pi r$$

$$\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt}$$

$$\frac{dA}{dt} = 2\pi r \times 8$$

when $r = 25$ cm

$$\frac{dA}{dt} = 2\pi \times 25 \times 8$$

$$= 1257 \text{ cm}^2 \text{ s}^{-1}$$

MEI Core 3 Chain Rule - Rates of Change Questions Jan 05 - May 09

1 Fig. 4 shows a cone. The angle between the axis and the slant edge is 30°. Water is poured into the cone at a constant rate of 2 cm^3 per second. At time t seconds, the radius of the water surface is r cm and the volume of water in the cone is $V\,cm^3$.

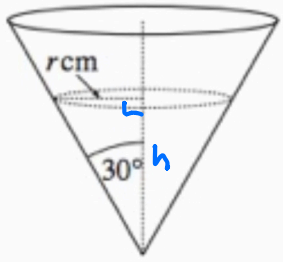

Fig. 4

$\tan 30° = \frac{r}{h}$

$h \tan 30 = r$

$\frac{h}{\sqrt{3}} = r$

$h = \sqrt{3}\, r$

(i) Write down the value of $\frac{dV}{dt}$. [1]

(ii) Show that $V = \frac{\sqrt{3}}{3}\pi r^3$, and find $\frac{dV}{dr}$. [3]

[You may assume that the volume of a cone of height h and radius r is $\frac{1}{3}\pi r^2 h$.]

(iii) Use the results of parts **(i)** and **(ii)** to find the value of $\frac{dr}{dt}$ when $r = 2$. [3]

i) $\frac{dV}{dt} = 2 \text{ cm}^3 \text{s}^{-1}$

ii) $V = \frac{1}{3}\pi r^2 h \Rightarrow V = \frac{1}{3}\pi r^2 \times \sqrt{3}\, r$

$V = \frac{\sqrt{3}}{3}\pi r^3$

$\frac{dV}{dr} = \sqrt{3}\pi r^2$

iii) $\frac{dr}{dt} = \frac{dr}{dV} \times \frac{dV}{dt}$

$= \frac{1}{\frac{dV}{dr}} \times \frac{dV}{dt}$

$$\frac{dr}{dt} = \frac{1}{\sqrt{3}\pi r^2} \times 2$$

when $r = 2$ $\quad \frac{dr}{dt} = \frac{1}{\sqrt{3}\pi \times 2^2} \times 2 = 0.092 \text{ cm s}^{-1}$

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2 Fig. 4 is a diagram of a garden pond.

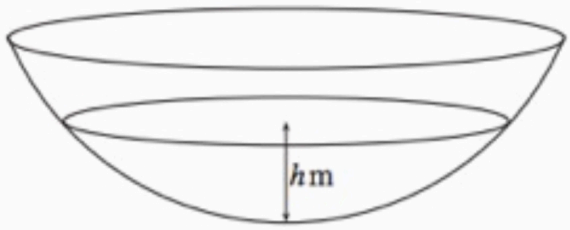

Fig. 4

The volume $V\,\text{m}^3$ of water in the pond when the depth is h metres is given by

$$V = \tfrac{1}{3}\pi h^2(3 - h).$$

(i) Find $\frac{dV}{dh}$. [2]

Water is poured into the pond at the rate of $0.02\,\text{m}^3$ per minute.

(ii) Find the value of $\frac{dh}{dt}$ when $h = 0.4$. [4]

i) $V = \frac{1}{3}\pi h^2(3 - h)$

$V = \pi h^2 - \frac{1}{3}\pi h^3$

$\frac{dV}{dh} = 2\pi h - \pi h^2$

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ii) $\frac{dV}{dt} = 0.02 \text{ m}^3/\text{min}$

$$\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt}$$

$$= \frac{1}{\frac{dV}{dh}} \times \frac{dV}{dt}$$

$$= \frac{1}{2\pi h - \pi h^2} \times 0.02$$

when $h = 0.4m$

$$\frac{dh}{dt} = \frac{1}{(2\pi \times 0.4 - \pi \times 0.4^2)} \times 0.02$$

$$= 0.0095 \text{ m/min}$$

Exercise 9J

1) $A = \frac{1}{4}\pi r^2$, $\frac{dr}{dt} = 6$ Find $\frac{dA}{dt}$ when $r = 2$

$$\frac{dA}{dr} = \frac{1}{2}\pi r$$

$$\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt}$$

$$= \frac{1}{2}\pi r \times 6$$

$$= 3\pi r$$

when $r = 2$, $\frac{dA}{dt} = 3\pi \times 2 = 6\pi = 18.8$

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3) $r = 1 + 3\cos\theta$, $\frac{d\theta}{dt} = 3$ Find $\frac{dr}{dt}$ when $\theta = \frac{\pi}{6}$

$$\frac{dr}{dt} = \frac{dr}{d\theta} \times \frac{d\theta}{dt} \qquad \frac{dr}{d\theta} = -3\sin\theta$$

$$\frac{dr}{dt} = -3\sin\theta \times 3 = -9\sin\theta$$

when $\theta = \frac{\pi}{6}$, $\frac{dr}{dt} = -9\sin\frac{\pi}{6} = -4.5$

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5) Let population $= P$

$$\frac{dP}{dt} = kP$$

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7) $$\frac{dV}{dt} = \underset{\text{in}}{30} - \underset{\text{out}}{\frac{2}{15}V}$$

$$15\frac{dV}{dt} = 450 - 2V$$

$$2V - 450 = -15\frac{dV}{dt}$$

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9) $$\frac{dx}{dt} = \frac{k}{x^2}$$

11) $\frac{dV}{dt} = -4.5 \text{ cm}^3 \text{s}^{-1}$

a) Find $\frac{dL}{dt}$ when $V = 100 \text{ cm}^3$

$V = L^3 \qquad \frac{dV}{dL} = 3L^2$

$$\frac{dL}{dt} = \frac{dL}{dV} \times \frac{dV}{dt}$$

$$= \frac{1}{\frac{dV}{dL}} \times \frac{dV}{dt}$$

$$\frac{dL}{dt} = \frac{1}{3L^2} \times -4.5$$

$V = 100$

$L = \sqrt[3]{100}$

when $L = \sqrt[3]{100}$

$$\frac{dL}{dt} = \frac{1}{3 \times 100^{2/3}} \times -4.5 = -0.070$$

side decreasing by 0.07 cm s^{-1}

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b) Find V when $\frac{dL}{dt} = -0.2 \text{ cm s}^{-1}$

$$\frac{dL}{dt} = \frac{1}{3L^2} \times -4.5$$

$$-0.2 = \frac{1}{3L^2} \times -4.5$$

$$3L^2 = \frac{-4.5}{-0.2}$$

$$L^2 = \frac{-4.5}{-0.2 \times 3}$$

$$L^2 = \frac{15}{2}$$

$$V = (L^2)^{3/2} = \left(\frac{15}{2}\right)^{3/2} = 20.5 \text{ cm}^3$$

Hwk Exercise 9J

Even numbers 2 to 10
