

1. There were 2100 tonnes of wheat harvested on a farm during 2017.

The mass of wheat harvested during each subsequent year is expected to increase by 1.2% per year.

- (a) Find the total mass of wheat expected to be harvested from 2017 to 2030 inclusive, giving your answer to 3 significant figures. (2)

Each year it costs

- £5.15 per tonne to harvest the first 2000 tonnes of wheat
- £6.45 per tonne to harvest wheat in excess of 2000 tonnes

- (b) Use this information to find the expected cost of harvesting the wheat from 2017 to 2030 inclusive. Give your answer to the nearest £1000 (3)

$$a) \quad GP \quad a = 2100, \quad r = 1.012, \quad n = 14$$

$$S_n = \frac{a(r^n - 1)}{r - 1} = \frac{2100(1.012^{14} - 1)}{1.012 - 1}$$

$$S_n = 31807 \text{ tonnes}$$

$$= 31800 \text{ tonnes to 3 s.f.}$$

$$b) \quad 14 \times 2000 = 28000 \text{ tonnes @ } \pounds 5.15$$

$$3807 \text{ tonnes @ } \pounds 6.45$$

$$\text{Total cost } \pounds 168,755$$

$$= \pounds 169,000 \text{ to nearest } \pounds 1000$$



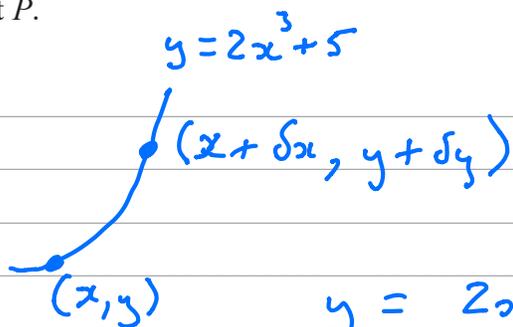


2. The curve  $C$  has equation

$$y = 2x^3 + 5$$

The curve  $C$  passes through the point  $P(1, 7)$ .

Use differentiation from first principles to find the value of the gradient of the tangent to  $C$  at  $P$ .

$$y = 2x^3 + 5 \quad (5)$$


$$y = 2x^3 + 5 \quad (1)$$

$$y + \delta y = 2(x + \delta x)^3 + 5$$

$$y + \delta y = 2(x^3 + 3x^2\delta x + 3x\delta x^2 + \delta x^3) + 5 \quad (2)$$

$$(2) - (1) \quad \delta y = 6x^2\delta x + 6x\delta x^2 + 2\delta x^3$$

$$\frac{\delta y}{\delta x} = 6x^2 + 6x\delta x + 2\delta x^2$$

$$\text{Gradient } \frac{dy}{dx} = \lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = 6x^2 + 0 + 0$$

$$\frac{dy}{dx} = 6x^2$$

$$\text{At } (1, 7) \quad \frac{dy}{dx} = 6(1)^2 = 6$$


---





3. The function  $f$  is defined by

$$f: x \mapsto \frac{3x-5}{x+1}, \quad x \in \mathbb{R}, \quad x \neq -1$$

(a) Find  $f^{-1}(x)$ . (3)

(b) Show that

$$ff(x) = \frac{x+a}{x-1}, \quad x \in \mathbb{R}, \quad x \neq \pm 1$$

where  $a$  is an integer to be found. (4)

The function  $g$  is defined by

$$g: x \mapsto x^2 - 3x, \quad x \in \mathbb{R}, \quad 0 \leq x \leq 5$$

(c) Find the value of  $fg(2)$ . (2)

(d) Find the range of  $g$ . (3)

(e) Explain why the function  $g$  does not have an inverse. (1)

a) Let  $y = \frac{3x-5}{x+1}$

swap variables  $x = \frac{3y-5}{y+1}$

$$xy + x = 3y - 5$$

$$x + 5 = 3y - xy$$

$$x + 5 = y(3-x)$$

$$\frac{x+5}{3-x} = y$$

$$f^{-1}(x) = \frac{x+5}{3-x} \quad \begin{matrix} x \in \mathbb{R} \\ x \neq 3 \end{matrix}$$



Question 3 continued

$$b) \quad f f(x) = f\left(\frac{3x-5}{x+1}\right)$$

$$= \frac{3\left(\frac{3x-5}{x+1}\right) - 5}{\left(\frac{3x-5}{x+1}\right) + 1}$$

$$= \frac{3(3x-5) - 5(x+1)}{3x-5 + x+1}$$

$$= \frac{9x - 15 - 5x - 5}{4x - 4}$$

$$= \frac{4x - 20}{4(x-1)}$$

$$= \frac{4(x-5)}{4(x-1)}$$

$$= \frac{x-5}{x-1}$$

so  $a = -5$ 

$$c) \quad f(x) = \frac{3x-5}{x+1} \quad g(x) = x^2 - 3x$$

$$f g(z) = f(z^2 - 3(z))$$

$$= f(-z)$$

(Total for Question 3 is 13 marks)



Question 3 continued

$$= \frac{3(-2) - 5}{-2 + 1}$$

$$= \frac{-11}{-1} = 11$$

d)

$$g(x) = x^2 - 3x$$
$$= \left(x - \frac{3}{2}\right)^2 - \frac{9}{4}$$

$$g(0) = 0$$

$$g\left(\frac{3}{2}\right) = -\frac{9}{4}$$

$$g(5) = 5^2 - 3(5) = 10$$

Range

$$-\frac{9}{4} \leq g(x) \leq 10$$

e)  $g$  is not a 1 to 1 mapping

eg

$$g(1) = 1 - 3(1) = -2$$

$$g(2) = 2^2 - 3(2) = -2$$

(Total for Question 3 is 13 marks)



4. Show that

$$\int_0^{\frac{\pi}{2}} \frac{\sin 2\theta}{1 + \cos \theta} d\theta = 2 - 2\ln 2 \quad (7)$$

$$\int_0^{\frac{\pi}{2}} \frac{\sin 2\theta}{1 + \cos \theta} d\theta = \int_0^{\frac{\pi}{2}} \frac{2 \sin \theta \cos \theta}{1 + \cos \theta} d\theta$$

$$\text{Let } u = 1 + \cos \theta \\ \Rightarrow \frac{du}{d\theta} = -\sin \theta$$

$$= \int_2^1 \frac{2(u-1)(-du)}{u}$$

$$du = -\sin \theta d\theta$$

$$-du = \sin \theta d\theta$$

$$= 2 \int_1^2 \frac{(u-1)}{u} du$$

$$\text{Also } \theta = 0 \Rightarrow u = 2 \\ \theta = \frac{\pi}{2} \Rightarrow u = 1$$

$$= 2 \int_1^2 \left(1 - \frac{1}{u}\right) du$$

$$= 2 \left[ u - \ln u \right]_1^2$$

$$= 2 \left[ (2 - \ln 2) - (1 - \ln 1) \right]$$

$$= 2 \left[ 1 - \ln 2 \right]$$

$$= 2 - 2\ln 2$$



**Question 4 continued**

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA

**(Total for Question 4 is 7 marks)**



5. (a) Express  $2 \sin \theta - 1.5 \cos \theta$  in the form  $R \sin(\theta - \alpha)$ , where  $R > 0$  and  $0 < \alpha < \frac{\pi}{2}$

State the value of  $R$  and give the value of  $\alpha$  to 4 decimal places.

(3)

Tom models the depth of water,  $D$  metres, at Southview harbour on 18th October 2017 by the formula

$$D = 6 + 2 \sin\left(\frac{4\pi t}{25}\right) - 1.5 \cos\left(\frac{4\pi t}{25}\right), \quad 0 \leq t \leq 24$$

where  $t$  is the time, in hours, after 00:00 hours on 18th October 2017.

Use Tom's model to

- (b) find the depth of water at 00:00 hours on 18th October 2017, (1)
- (c) find the maximum depth of water, (1)
- (d) find the time, in the afternoon, when the maximum depth of water occurs. Give your answer to the nearest minute. (3)

Tom's model is supported by measurements of  $D$  taken at regular intervals on 18th October 2017. Jolene attempts to use a similar model in order to model the depth of water at Southview harbour on 19th October 2017.

Jolene models the depth of water,  $H$  metres, at Southview harbour on 19th October 2017 by the formula

$$H = 6 + 2 \sin\left(\frac{4\pi x}{25}\right) - 1.5 \cos\left(\frac{4\pi x}{25}\right), \quad 0 \leq x \leq 24$$

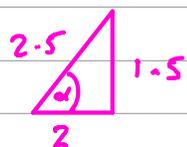
where  $x$  is the time, in hours, after 00:00 hours on 19th October 2017.

By considering the depth of water at 00:00 hours on 19th October 2017 for both models,

- (e) (i) explain why Jolene's model is not correct, (3)
- (ii) hence find a suitable model for  $H$  in terms of  $x$ .

$$a) \quad 2 \sin \theta - 1.5 \cos \theta$$

$$= 2.5 \sin(\theta - 0.6435)$$



$$\alpha = \tan^{-1}\left(\frac{1.5}{2}\right) = 0.6435 \quad R = 2.5$$



Question 5 continued

$$b) \quad D = 6 + 2 \sin 0 - 1.5 \cos 0$$

$$D = 6 + 0 - 1.5$$

$$D = 4.5 \text{ m}$$

---

$$c) \quad D = 6 + 2.5 \sin\left(\frac{4\pi t}{25} - 0.6435\right)$$

$$D_{\max} = 6 + 2.5 = 8.5 \text{ m}$$

---

$$d) \quad D_{\max} \text{ when } \frac{4\pi t}{25} - 0.6435 = \frac{\pi}{2}$$

$$4\pi t = 25\left(\frac{\pi}{2} + 0.6435\right)$$

$$t = \frac{25}{4\pi} \left(\frac{\pi}{2} + 0.6435\right)$$

$$t = 4.405 \text{ hrs} \quad \text{X too early}$$

$$\frac{4\pi t}{25} - 0.6435 = \frac{5\pi}{2}$$

$$t = \frac{25}{4\pi} \left(\frac{5\pi}{2} + 0.6435\right)$$

$$t = 16.9052 \text{ hrs}$$

$$t = 16:54 \text{ hrs}$$

---



Question 5 continued

e) Depth at 00:00 on 19<sup>th</sup> Oct

Jolene 4.5 m

$$\text{Tom } 6 + 2.5 \sin\left(\frac{4\pi \times 24}{25} - 0.6435\right)$$

$$= 3.72 \text{ m}$$

4.5  $\neq$  3.72 so Jolene's model

not correct

Suitable Model

$$H = 6 + 2.5 \sin\left(\frac{4\pi(x+24)}{25} - 0.6435\right)$$

---



**Question 5 continued**

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA

**(Total for Question 5 is 11 marks)**



6.

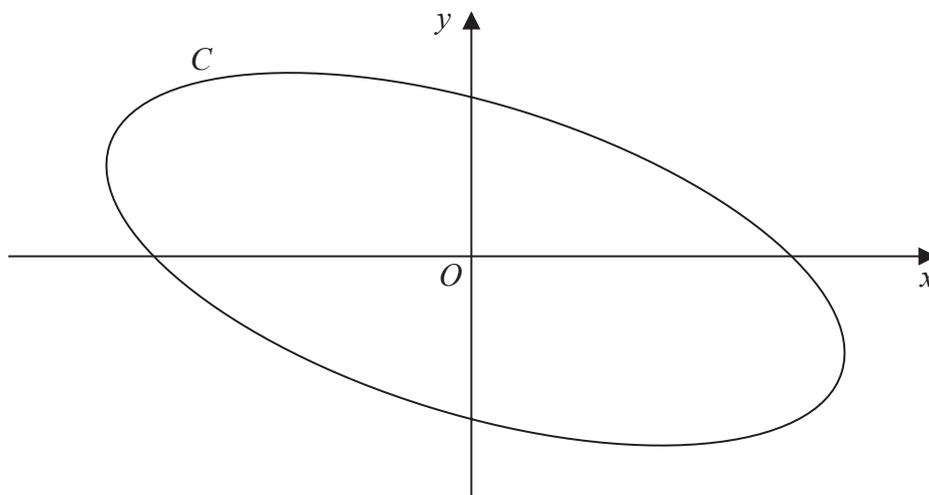


Figure 6

Figure 6 shows a sketch of the curve  $C$  with parametric equations

$$x = 4 \cos\left(t + \frac{\pi}{6}\right), \quad y = 2 \sin t, \quad 0 < t \leq 2\pi$$

Show that a Cartesian equation of  $C$  can be written in the form

$$(x + y)^2 + ay^2 = b$$

where  $a$  and  $b$  are integers to be found.

(5)

$$x = 4 \cos\left(t + \frac{\pi}{6}\right)$$

$$x = 4 \left( \cos t \cos \frac{\pi}{6} - \sin t \sin \frac{\pi}{6} \right)$$

$$x = 4 \left( \frac{\sqrt{3}}{2} \cos t - \frac{1}{2} \sin t \right)$$

$$\underline{x = 2\sqrt{3} \cos t - 2 \sin t}$$

$$y = 2 \sin t$$

$$\frac{y}{2} = \sin t$$

$$\cos t = \sqrt{1 - \sin^2 t} = \sqrt{1 - \frac{y^2}{4}}$$



Question 6 continued

$$\Rightarrow x = 2\sqrt{3} \sqrt{1 - \frac{y^2}{4}} - y$$

$$x + y = 2\sqrt{3} \sqrt{1 - \frac{y^2}{4}}$$

$$(x + y)^2 = 12 \left(1 - \frac{y^2}{4}\right)$$

$$(x + y)^2 = 12 - 3y^2$$

$$(x + y)^2 + 3y^2 = 12$$

---

$$a = 3$$

$$b = 12$$

---





7. (a) Sketch the graph with equation

$$y = |2x - 5|$$

stating the coordinates of any points where the graph cuts or meets the coordinate axes.

(2)

- (b) Find the values of  $x$  which satisfy

$$|2x - 5| > 7$$

(2)

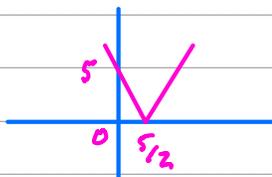
- (c) Find the values of  $x$  which satisfy

$$|2x - 5| > x - \frac{5}{2}$$

Write your answer in set notation.

(2)

a)



Cuts y-axis at  $(0, 5)$

Meets x-axis at  $(\frac{5}{2}, 0)$

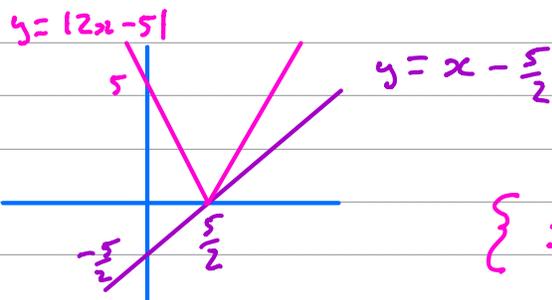
b)

$$\begin{aligned} \text{Either } 2x - 5 &> 7 \\ 2x &> 12 \\ x &> 6 \end{aligned}$$

$$\begin{aligned} \text{or } 2x - 5 &< -7 \\ 2x &< -2 \\ x &< -1 \end{aligned}$$

$$\text{Either } x < -1 \text{ or } x > 6$$

c)



$$\left\{ x \in \mathbb{R} \setminus \left\{ \frac{5}{2} \right\} \right\}$$

ie all values on number line except  $\frac{5}{2}$





8. The line  $l$  has equation

$$3x - 2y = k$$

①

where  $k$  is a real constant.

Given that the line  $l$  intersects the curve with equation

$$y = 2x^2 - 5$$

②

at two distinct points, find the range of possible values for  $k$ .

(5)

Sub for  $y$  in ①

$$3x - 2(2x^2 - 5) = k$$

$$3x - 4x^2 + 10 = k$$

$$0 = 4x^2 - 3x + (k - 10)$$

for 2 distinct points  $b^2 > 4ac$

$$(-3)^2 > 4 \times 4 \times (k - 10)$$

$$9 > 16k - 160$$

$$169 > 16k$$

$$\frac{169}{16} > k$$

$$k < \frac{169}{16}$$



**Question 8 continued**

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA

**(Total for Question 8 is 5 marks)**



9.

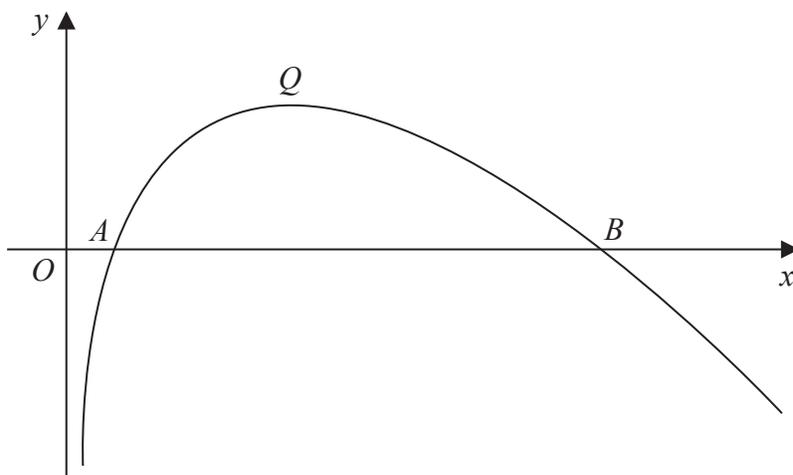


Figure 2

Figure 2 shows a sketch of the curve with equation  $y = f(x)$ , where

$$f(x) = (8 - x) \ln x, \quad x > 0$$

The curve cuts the  $x$ -axis at the points  $A$  and  $B$  and has a maximum turning point at  $Q$ , as shown in Figure 2.

(a) Find the  $x$  coordinate of  $A$  and the  $x$  coordinate of  $B$ .

(1)

(b) Show that the  $x$  coordinate of  $Q$  satisfies

$$x = \frac{8}{1 + \ln x}$$

(4)

(c) Show that the  $x$  coordinate of  $Q$  lies between 3.5 and 3.6

(2)

(d) Use the iterative formula

$$x_{n+1} = \frac{8}{1 + \ln x_n} \quad n \in \mathbb{N}$$

with  $x_1 = 3.5$  to

(i) find the value of  $x_5$  to 4 decimal places,

(ii) find the  $x$  coordinate of  $Q$  accurate to 2 decimal places.

(2)

a)  $A$  &  $B$  and  $B$   $f(x) = 0$

$$(8 - x) \ln x = 0$$

$$\Rightarrow \begin{array}{l} 8 - x = 0 \\ x = 8 \end{array} \quad \text{or} \quad \begin{array}{l} \ln x = 0 \\ x = 1 \end{array}$$



Question 9 continued

$$\text{At } A, x = 1$$

$$\text{At } B, x = 8$$

b)

$$f(x) = (8-x) \ln x$$

$$f'(x) = (8-x) \frac{1}{x} + (-1) \ln x$$

$$f'(x) = \frac{8-x}{x} - \ln x$$

$$f'(x) = \frac{8-x-x \ln x}{x}$$

$$\text{At } Q, f'(x) = 0$$

$$\Rightarrow 8-x-x \ln x = 0$$

$$8 = x + x \ln x$$

$$8 = x(1 + \ln x)$$

$$\frac{8}{1 + \ln x} = x$$

$$x = \frac{8}{1 + \ln x}$$

$$\hookrightarrow f'(3.5) = \frac{8 - 3.5 - 3.5 \ln 3.5}{3.5} = 0.033$$

$$f'(3.6) = \frac{8 - 3.6 - 3.6 \ln 3.6}{3.6} = -0.059$$

(Total for Question 9 is 9 marks)



Question 9 continued

$f'(x)$  is continuous between  $x = 3.5$

and  $x = 3.6$ . Since  $f'(3.5) > 0$  and  $f'(3.6) < 0$

$f'(x) = 0$  for some value between  $x = 3.5$  and

$x = 3.6$ .  $\therefore$   $x$ -coord of  $Q$  lies between 3.5 and 3.6

---

c) i)  $x_{n+1} = \frac{8}{1 + \ln x_n}$        $x_2 = \frac{8}{1 + \ln 3.5} = 3.5512$

$x_3 = \frac{8}{1 + \ln(\text{Ans})} = 3.5286$

$x_4 = 3.5385$

$x_5 = 3.5340$  to 4 d.p.

---

ii)  $x$ -coord of  $Q = 3.54$  to 2 d.p.

---

$x_6 = 3.5360$

$x_7 = 3.5351$

Last part unfair - you would need to evaluate

as far as  $x_7$  before two iterations agreed

to 2 decimal places

---

(Total for Question 9 is 9 marks)



10. A circle with centre  $A(3, -1)$  passes through the point  $P(-9, 8)$  and the point  $Q(15, -10)$

(a) Show that  $PQ$  is a diameter of the circle. (2)

(b) Find an equation for the circle. (3)

$$A(3, -1)$$

$$P(-9, 8)$$

$$Q(15, -10)$$

A point  $R$  also lies on the circle.

Given that the length of the chord  $PR$  is 20 units,

(c) find the length of the shortest distance from  $A$  to the chord  $PR$ .  
Give your answer as a surd in its simplest form. (2)

(d) Find the size of angle  $ARQ$ , giving your answer to the nearest 0.1 of a degree. (2)

a)

$$\text{Gradient } AP = \frac{8 - (-1)}{-9 - 3} = \frac{9}{-12} = -\frac{3}{4}$$

$$\text{Gradient } AQ = \frac{-10 - (-1)}{15 - 3} = \frac{-9}{12} = -\frac{3}{4}$$

Gradients the same so  $PQ$  is a straight line

through centre  $A$ ,  $P, Q$  both on circle

so  $PQ$  is a diameter.

$$\text{b) radius } AP = \sqrt{(3 - (-9))^2 + (-1 - 8)^2}$$

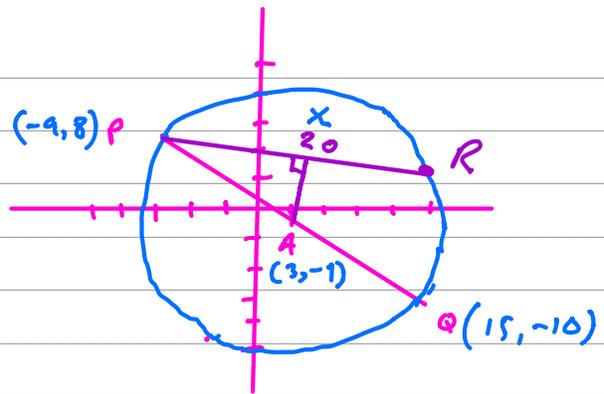
$$\text{radius} = \sqrt{12^2 + (-9)^2} = \sqrt{225} = 15$$

$$\text{Circle } (x - 3)^2 + (y + 1)^2 = 15^2$$



Question 10 continued

c) Let  $R(x, y)$



Shortest distance  $AX$  is  $\perp$  bisector of chord  $PR$

Pythagoras  $AX^2 + PX^2 = AP^2$

$$AX^2 + 10^2 = 15^2$$

$$AX^2 = 15^2 - 10^2$$

$$AX^2 = 225 - 100 = 125$$

$$AX = \sqrt{125}$$

$$AX = 5\sqrt{5}$$

d)  $\angle PRQ = 90^\circ$  ( $\angle$  in semi-circle)

$$\angle ARP = \cos^{-1}\left(\frac{10}{15}\right) = 48.2^\circ \quad (\angle \text{ in } \triangle XRA)$$

$$\angle ARQ = \angle PQR - \angle ARP$$

$$= 90^\circ - 48.2^\circ$$

$$= 41.8^\circ$$

(Total for Question 10 is 9 marks)



11. The second, third and fourth terms of an arithmetic sequence are  $2k$ ,  $5k-10$  and  $7k-14$  respectively, where  $k$  is a constant.

Show that the sum of the first  $n$  terms of the sequence is a square number.

(5)

AP

$$\text{1st} \quad a \quad (1)$$

$$\text{2nd} \quad a+d = 2k \quad (2)$$

$$\text{3rd} \quad a+2d = 5k-10 \quad (3)$$

$$\text{4th} \quad a+3d = 7k-14 \quad (4)$$

$$(3) - (2) \quad d = 3k - 10 \quad (5)$$

$$(4) - (3) \quad d = 2k - 4 \quad (6)$$

$$(5) - (6) \quad 0 = k - 6$$

$$\Rightarrow k = 6$$

Sub for  $k$  in (6)

$$d = 2(6) - 4$$

$$\underline{d = 8}$$

Sub for  $d, k$  in (2)

$$a + d = 2k$$

$$a + 8 = 2(6)$$

$$\underline{a = 4}$$



Question 11 continued

$$S_n = \frac{n}{2} (2a + (n-1)d)$$

$$S_n = \frac{n}{2} (2(4) + 8(n-1))$$

$$S_n = \frac{n}{2} (8 + 8n - 8)$$

$$S_n = \frac{n}{2} (8n)$$

$$S_n = 4n^2$$

$$S_n = (2n)^2$$

This is a square number since

$2n$  is an integer

---

(Total for Question 11 is 5 marks)



12. A curve  $C$  is given by the equation

$$\sin x + \cos y = 0.5 \quad -\frac{\pi}{2} \leq x < \frac{3\pi}{2}, -\pi < y < \pi$$

A point  $P$  lies on  $C$ .

The tangent to  $C$  at the point  $P$  is parallel to the  $x$ -axis.

Find the exact coordinates of all possible points  $P$ , justifying your answer.  
(Solutions based entirely on graphical or numerical methods are not acceptable.)

(7)

$$\sin x + \cos y = 0.5$$

diff wrt  $x$

$$\cos x - \sin y \frac{dy}{dx} = 0$$

$$\cos x = \sin y \frac{dy}{dx}$$

$$\frac{\cos x}{\sin y} = \frac{dy}{dx}$$

Gradient at  $P$  is 0 since  $tgt$  parallel to  $x$ -axis

$$\Rightarrow \frac{dy}{dx} = 0$$

$$\Rightarrow \cos x = 0$$

$$\Rightarrow x = -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}$$

$$-\frac{\pi}{2} \leq x < \frac{3\pi}{2} \quad \text{so} \quad x = -\frac{\pi}{2}, \frac{\pi}{2}$$



Question 12 continued

When  $x = -\frac{\pi}{2}$

$$\sin\left(-\frac{\pi}{2}\right) + \cos y = 0.5$$

$$-1 + \cos y = 0.5$$

$$\cos y = 1.5 \quad \text{no solution}$$

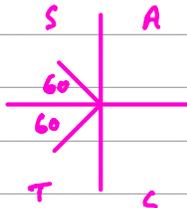
When  $x = \frac{\pi}{2}$

$$\sin\left(\frac{\pi}{2}\right) + \cos y = 0.5$$

$$1 + \cos y = 0.5$$

$$\cos y = -0.5$$

$$\Rightarrow y = \frac{2\pi}{3}, -\frac{2\pi}{3}$$



Possible coordinates of P

$$\left(\frac{\pi}{2}, \frac{2\pi}{3}\right) \quad \text{and} \quad \left(\frac{\pi}{2}, -\frac{2\pi}{3}\right)$$

(Total for Question 12 is 7 marks)



13. (a) Show that

$$\operatorname{cosec} 2x + \cot 2x \equiv \cot x, \quad x \neq 90n^\circ, n \in \mathbb{Z} \quad (5)$$

(b) Hence, or otherwise, solve, for  $0 \leq \theta < 180^\circ$ ,

$$\operatorname{cosec}(4\theta + 10^\circ) + \cot(4\theta + 10^\circ) = \sqrt{3}$$

You must show your working.

(Solutions based entirely on graphical or numerical methods are not acceptable.)

(5)

a)

$$\operatorname{cosec} 2x + \cot 2x$$

$$\equiv \frac{1}{\sin 2x} + \frac{\cos 2x}{\sin 2x}$$

$$\equiv \frac{1 + \cos 2x}{\sin 2x}$$

$$\equiv \frac{2 \cos^2 x}{2 \sin x \cos x}$$

$$\equiv \frac{\cos x}{\sin x}$$

$$\equiv \cot x \quad \text{as required}$$



Question 13 continued

$$b) \quad \operatorname{cosec}(4\theta + 10^\circ) + \cot(4\theta + 10^\circ) = \sqrt{3}$$

$$\Rightarrow \cot(2\theta + 5^\circ) = \sqrt{3}$$

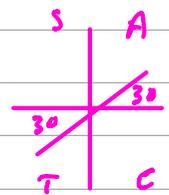
$$\Rightarrow \tan(2\theta + 5^\circ) = \frac{1}{\sqrt{3}}$$

$$\Rightarrow 2\theta + 5^\circ = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

$$\Rightarrow 2\theta + 5^\circ = 30^\circ, 210^\circ$$

$$\Rightarrow 2\theta = 25^\circ, 205^\circ$$

$$\Rightarrow \theta = 12.5^\circ, 102.5^\circ$$



(Total for Question 13 is 10 marks)

