

Vectors, 11 Mixed Exercise

1 a $\mathbf{R} = -3\mathbf{i} + 7\mathbf{j} + \mathbf{i} - \mathbf{j}$
 $= -2\mathbf{i} + 6\mathbf{j}$

$$|\mathbf{R}| = \sqrt{2^2 + 6^2}$$

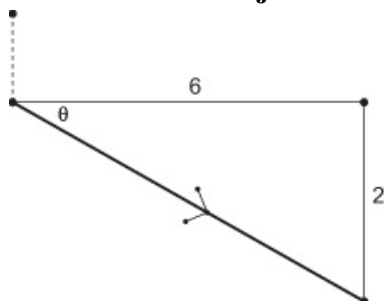
$$= \sqrt{40}$$

$$= 2\sqrt{10}$$

The magnitude of \mathbf{R} is $2\sqrt{10}$ N

b $\tan \theta = \frac{1}{3}$
 $\theta = \tan^{-1} \frac{1}{3}$
 $= 18^\circ$ (nearest degree)

2 a (Path of S) = $(4\mathbf{i} - 6\mathbf{j}) - (-2\mathbf{i} - 4\mathbf{j})$
 $= 6\mathbf{i} - 2\mathbf{j}$



$$\tan \theta = \frac{1}{3} \Rightarrow \theta = 18.43\dots^\circ$$

Bearing = $90^\circ + \theta = 108^\circ$

b Expressing velocity, \mathbf{v} , in km h^{-1} :
 $\mathbf{v} = (6\mathbf{i} - 2\mathbf{j}) \times \frac{60}{40}$

$$\mathbf{v} = 9\mathbf{i} - 3\mathbf{j}$$

Then the speed is:

$$\sqrt{9^2 + (-3)^2} = \sqrt{90} = 3\sqrt{10}$$

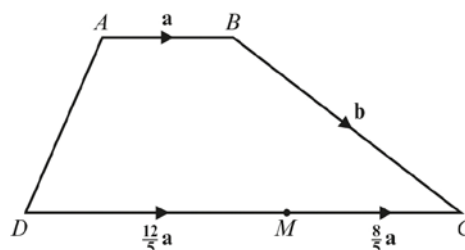
$$= 9.49 \text{ km h}^{-1} \text{ (3 s.f.)}$$

3 a Speed = $\sqrt{9^2 + 4^2}$
 $= \sqrt{97}$
 $= 9.85 \text{ m s}^{-1}$ (3 s.f.)

b Distance = speed \times time
 $= \sqrt{97} \times 6$
 $= 59.1 \text{ m}$

c This model becomes less accurate as t increases because it ignores friction and air resistance.

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a $\overrightarrow{AM} = \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CM}$
 $= \mathbf{a} + \mathbf{b} - \frac{8}{5}\mathbf{a}$
 $= \mathbf{b} - \frac{3}{5}\mathbf{a}$

b $\overrightarrow{BD} = \overrightarrow{BC} + \overrightarrow{CD}$
 $= \mathbf{b} - 4\mathbf{a}$

c $\overrightarrow{MB} = \overrightarrow{MC} + \overrightarrow{CB}$
 $= \frac{8}{5}\mathbf{a} - \mathbf{b}$

d $\overrightarrow{DA} = \overrightarrow{DC} + \overrightarrow{CB} + \overrightarrow{BA}$
 $= 4\mathbf{a} - \mathbf{b} - \mathbf{a}$
 $= 3\mathbf{a} - \mathbf{b}$

5 As the vectors are parallel
 $5\mathbf{a} + k\mathbf{b} = \frac{5}{8}(8\mathbf{a} + 2\mathbf{b})$
 $k = \frac{5}{8} \times 2$
 $= 1.25$

6 a $\mathbf{a} + \mathbf{b} + \mathbf{c} = \begin{pmatrix} 7 \\ 4 \end{pmatrix} + \begin{pmatrix} 10 \\ -2 \end{pmatrix} + \begin{pmatrix} -5 \\ -3 \end{pmatrix}$
 $= \begin{pmatrix} 12 \\ -1 \end{pmatrix}$

b $\mathbf{a} - 2\mathbf{b} + \mathbf{c} = \begin{pmatrix} 7 \\ 4 \end{pmatrix} - 2\begin{pmatrix} 10 \\ -2 \end{pmatrix} + \begin{pmatrix} -5 \\ -3 \end{pmatrix}$
 $= \begin{pmatrix} -18 \\ 5 \end{pmatrix}$

c $2\mathbf{a} + 2\mathbf{b} - 3\mathbf{c} = 2\begin{pmatrix} 7 \\ 4 \end{pmatrix} + 2\begin{pmatrix} 10 \\ -2 \end{pmatrix} - 3\begin{pmatrix} -5 \\ -3 \end{pmatrix}$
 $= \begin{pmatrix} 49 \\ 13 \end{pmatrix}$

$$\begin{aligned}
 7 \text{ a } \quad \overrightarrow{BC} &= \overrightarrow{BA} + \overrightarrow{AC} \\
 &= -(3\mathbf{i} + 5\mathbf{j}) + 6\mathbf{i} + 3\mathbf{j} \\
 &= 3\mathbf{i} - 2\mathbf{j}
 \end{aligned}$$

$$\begin{aligned}
 \text{b } \quad \tan x &= \frac{5}{3} \\
 x &= \tan^{-1} \frac{5}{3} \\
 &= 59.036\dots \\
 \tan y &= \frac{1}{2} \\
 y &= \tan^{-1} \frac{1}{2} \\
 &= 26.565\dots \\
 \angle BAC &= 59.036\dots - 26.565\dots \\
 &= 32.5^\circ \text{ (3 s.f.)}
 \end{aligned}$$

$$\begin{aligned}
 \text{c } \quad \text{Area} &= \frac{1}{2} bc \sin A \\
 b &= \sqrt{6^2 + 3^2} = \sqrt{45} \\
 c &= \sqrt{3^2 + 5^2} = \sqrt{34} \\
 \text{Area} &= \frac{1}{2} \times \sqrt{45} \times \sqrt{34} \times \sin 32.5^\circ \\
 &= 10.508\dots \\
 &= 10.5 \text{ units}^2 \text{ (3 s.f.)}
 \end{aligned}$$

$$\begin{aligned}
 8 \text{ a } \quad 4\mathbf{i} - 3\mathbf{j} + 2p\mathbf{i} - p\mathbf{j} &= \lambda(2\mathbf{i} - 3\mathbf{j}) \\
 (4 + 2p)\mathbf{i} - (3 + p)\mathbf{j} &= 2\lambda\mathbf{i} - 3\lambda\mathbf{j} \\
 \text{Equating coefficients:} \\
 4 + 2p &= 2\lambda \text{ and } 3 + p = 3\lambda \\
 \text{Solving simultaneously:} \\
 \text{Rearranging the } 3 + p = 3\lambda \\
 p &= 3\lambda - 3 \\
 \text{Using substitution:} \\
 4 + 2(3\lambda - 3) &= 2\lambda \\
 4 + 6\lambda - 6 &= 2\lambda \\
 4\lambda &= 2 \\
 \lambda &= \frac{1}{2} \\
 p &= -1.5
 \end{aligned}$$

$$\begin{aligned}
 \text{b } \quad \mathbf{a} + \mathbf{b} &= 4\mathbf{i} - 3\mathbf{j} - 3\mathbf{i} + 1.5\mathbf{j} \\
 &= \mathbf{i} - 1.5\mathbf{j}
 \end{aligned}$$

$$\begin{aligned}
 9 \text{ a } \text{ i } \quad \text{a unit vector is } \frac{\mathbf{a}}{|\mathbf{a}|} \\
 \mathbf{a} &= \begin{pmatrix} 8 \\ 15 \end{pmatrix} \\
 |\mathbf{a}| &= \sqrt{8^2 + 15^2} = \sqrt{289} = 17
 \end{aligned}$$

$$9 \text{ a } \text{ i } \quad \frac{\mathbf{a}}{|\mathbf{a}|} = \frac{1}{17} \begin{pmatrix} 8 \\ 15 \end{pmatrix} = \frac{1}{17} (8\mathbf{i} + 15\mathbf{j})$$

$$\begin{aligned}
 \text{ii } \quad \tan \theta &= \frac{15}{8} \\
 \theta &= \tan^{-1} \frac{15}{8} \\
 &= 61.9^\circ \text{ (3 s.f.) above}
 \end{aligned}$$

$$\text{b } \text{ i } \quad \text{a unit vector is } \frac{\mathbf{b}}{|\mathbf{b}|}$$

$$\begin{aligned}
 \mathbf{b} &= \begin{pmatrix} 24 \\ -7 \end{pmatrix} \\
 |\mathbf{b}| &= \sqrt{24^2 + 7^2} \\
 &= \sqrt{625} \\
 &= 25 \\
 \frac{\mathbf{b}}{|\mathbf{b}|} &= \frac{1}{25} \begin{pmatrix} 24 \\ -7 \end{pmatrix} \\
 &= \frac{1}{25} (24\mathbf{i} - 7\mathbf{j})
 \end{aligned}$$

$$\begin{aligned}
 \text{ii } \quad \tan \theta &= \frac{7}{24} \\
 \theta &= \tan^{-1} \frac{7}{24} \\
 &= 16.3^\circ \text{ (3 s.f.) below}
 \end{aligned}$$

$$\text{c } \text{ i } \quad \text{a unit vector is } \frac{\mathbf{c}}{|\mathbf{c}|}$$

$$\begin{aligned}
 \mathbf{c} &= \begin{pmatrix} -9 \\ 40 \end{pmatrix} \\
 |\mathbf{c}| &= \sqrt{9^2 + 40^2} \\
 &= \sqrt{1681} \\
 &= 41 \\
 \frac{\mathbf{c}}{|\mathbf{c}|} &= \frac{1}{41} \begin{pmatrix} -9 \\ 40 \end{pmatrix} \\
 &= \frac{1}{41} (-9\mathbf{i} + 40\mathbf{j})
 \end{aligned}$$

$$\begin{aligned}
 \text{ii } \quad \tan x &= \frac{40}{9} \\
 x &= \tan^{-1} \frac{40}{9} \\
 &= 77.3^\circ \text{ (3 s.f.)} \\
 \theta &= 180^\circ - 77.3^\circ \\
 &= 102.7^\circ \text{ above}
 \end{aligned}$$

9 d i a unit vector is $\frac{\mathbf{d}}{|\mathbf{d}|}$

$$\mathbf{d} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$$

$$|\mathbf{d}| = \sqrt{3^2 + 2^2} \\ = \sqrt{13}$$

$$\frac{\mathbf{d}}{|\mathbf{d}|} = \frac{1}{\sqrt{13}} \begin{pmatrix} 3 \\ -2 \end{pmatrix} \\ = \frac{1}{\sqrt{13}}(3\mathbf{i} - 2\mathbf{j})$$

ii $\tan \theta = \frac{2}{3}$

$$\theta = \tan^{-1} \frac{2}{3}$$

$$= 33.7^\circ \text{ (3 s.f.) below}$$

10 $\cos 55^\circ = \frac{p}{15}$
 $p = 15 \cos 55^\circ$
 $p = 8.6$

Using Pythagoras' theorem:

$$q = \sqrt{15^2 - 8.6^2} \\ = 12.3 \\ p = 8.6 \text{ and } q = 12.3$$

11 $|\mathbf{3i} - \mathbf{kj}| = \sqrt{3^2 + k^2}$
 $= \sqrt{9 + k^2}$
 $= 3\sqrt{5}$

$$\sqrt{9 + k^2} = \sqrt{45} \\ k^2 + 9 = 45 \\ k^2 = 36 \\ k = \pm 6$$

12 a $\overrightarrow{ON} = \overrightarrow{OA} + \overrightarrow{AN}$ or $\overrightarrow{ON} = \overrightarrow{OM} + \overrightarrow{MN}$
 $\frac{\overrightarrow{AB}}{AB} = \frac{\overrightarrow{AO} + \overrightarrow{OB}}{AB}$
 $= -\mathbf{a} + \mathbf{b}$
 $\overrightarrow{MN} = \lambda \mathbf{b}$

Using similar triangles:

$$\overrightarrow{AN} = \lambda(-\mathbf{a} + \mathbf{b})$$

Using $\overrightarrow{ON} = \overrightarrow{OA} + \overrightarrow{AN}$

$$\overrightarrow{ON} = \mathbf{a} + \lambda(-\mathbf{a} + \mathbf{b}) = (1 - \lambda)\mathbf{a} + \lambda\mathbf{b}$$

12 a Using $\overrightarrow{ON} = \overrightarrow{OM} + \overrightarrow{MN}$
 $\overrightarrow{ON} = \frac{3}{5}\mathbf{a} + \lambda\mathbf{b}$

$$(1 - \lambda)\mathbf{a} + \lambda\mathbf{b} = \frac{3}{5}\mathbf{a} + \lambda\mathbf{b}$$

Equating coefficients

$$1 - \lambda = \frac{3}{5}$$

$$\lambda = \frac{2}{5}$$

$$\overrightarrow{ON} = \frac{3}{5}\mathbf{a} + \frac{2}{5}\mathbf{b}$$

b $\overrightarrow{MN} = \lambda\mathbf{b}$
 $= \frac{2}{5}\mathbf{b}$

c $\overrightarrow{AN} = \frac{2}{5}(-\mathbf{a} + \mathbf{b})$

$$\overrightarrow{AB} = -\mathbf{a} + \mathbf{b}$$

Therefore, $AN : NB = 2 : 3$

13 a $\tan \theta = \frac{1}{3}$
 $\theta = \tan^{-1} \frac{1}{3}$
 $= 18.4^\circ \text{ below}$

b $4\mathbf{i} - 5\mathbf{j} + p\mathbf{i} + q\mathbf{j} = \lambda(3\mathbf{i} - \mathbf{j})$
 $(4 + p)\mathbf{i} + (q - 5)\mathbf{j} = 3\lambda\mathbf{i} - \lambda\mathbf{j}$
 $4 + p = 3\lambda$ and $q - 5 = -\lambda$

Multiplying the second equation by 3:

$$3q - 15 = -3\lambda$$

Solving simultaneously:

$$4 + p = -3q + 15$$

$$p + 3q = 11$$

c When $p = 2, \lambda = 2$.

$$\mathbf{R} = 2(3\mathbf{i} - \mathbf{j}) \\ = 6\mathbf{i} - 2\mathbf{j}$$

$$|\mathbf{R}| = \sqrt{6^2 + 2^2}$$

$$= \sqrt{40}$$

$$= 2\sqrt{10} \text{ N}$$

$$\begin{aligned}
 14 \quad \mathbf{v} - \mathbf{u} &= (15\mathbf{i} - 3\mathbf{j}) - (3\mathbf{i} + 4\mathbf{j}) \\
 &= 12\mathbf{i} - 7\mathbf{j} \\
 |\mathbf{a}| &= \frac{\sqrt{12^2 + 7^2}}{2} \\
 &= \frac{\sqrt{193}}{2}
 \end{aligned}$$

Challenge

$$y = 5 - \frac{5}{3}x$$

Using Pythagoras' theorem:

$$x^2 + y^2 = \frac{17}{2}$$

Solve the equations simultaneously.

Substitute $y = 5 - \frac{5}{3}x$ into $x^2 + y^2 = \frac{17}{2}$.

$$x^2 + \left(5 - \frac{5}{3}x\right)^2 = \frac{17}{2}$$

$$x^2 + 25 - \frac{50}{3}x + \frac{25}{9}x^2 - \frac{17}{2} = 0$$

$$18x^2 + 450 - 300x + 50x^2 - 153 = 0$$

$$68x^2 - 300x + 297 = 0$$

Using the quadratic formula:

$$x =$$

$$x = \frac{300 \pm \sqrt{9216}}{136}$$

$$x = \frac{300 \pm 96}{136}$$

$$x = \frac{99}{34} \text{ or } x = \frac{51}{34}$$

$$\text{When } x = \frac{99}{34}, y = \frac{5}{34}$$

$$\text{When } x = \frac{51}{34}, y = \frac{5}{2}$$

$$\overrightarrow{OB} = \frac{99}{34}\mathbf{i} + \frac{5}{34}\mathbf{j} \text{ or } \overrightarrow{OB} = \frac{51}{34}\mathbf{i} + \frac{5}{2}\mathbf{j}$$